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Portfolio Return Maximization using Robust Optimization and Directional Changes

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***Abstract*—Dynamic portfolio optimization is inherently challenging due to the complexity of asset price dynamics and forecasts. Robust optimization is proposed as an alternative that incorporates return and risk uncertainty in portfolio optimization. Directional change (DC) methods complement the standard, fixed time interval, and asset price data in terms of measuring the relationships and scaling laws between different types of events. DC methods can be extended for portfolio optimization using DC representations of assets and empirical scaling laws which indicate expected price changes and their duration. In this paper, we study a robust DC-based portfolio optimization (RDC) method, for returns maximization. The proposed method uses price signals from the DC representations of multiple assets for portfolio rebalancing and optimization, together with a robust portfolio optimization rule that maximizes portfolio returns under return uncertainty. We empirically study the effect of the robust DC-based portfolio optimization method with an application to 29 exchange-traded funds where each fund is a well-diversified asset with typically low-risk values. We compare the obtained portfolio results with benchmarks. The results indicate that the proposed method performs comparably to several benchmarks, and particularly improves a specific risk measure, maximum drawdown, in comparison to the benchmarks.**

***Index Terms*—Directional changes, financial portfolio optimization, robust optimization.**

I. Introduction

Portfolio optimization is an important task for individuals and institutions that aim at a high financial performance from their assets. The performance of an optimized portfolio is affected by general market dynamics as well as specific realizations of returns and correlations across different assets. Standard portfolio optimization techniques such as mean-variance optimization rely on historical or model-based estimates of returns and correlations to dynamically adapt portfolios. These methods typically do not incorporate the uncertainty of inputs and their results are shown to be very sensitive to small changes in inputs [1], [2].

Robust optimization [3]–[5] was developed and applied in portfolio optimization in order to incorporate the inherent randomness in expected returns and to improve the stability of portfolio results when the realized values of inputs deviate from the estimated ones within some given set. The performance of specific robust optimization methods depends on the definition of the uncertain variable, the defined uncertainty set, and calibration of the uncertainty parameter [6]. In general, robust portfolio optimization is shown to be valuable in several applications [7]–[9], but to the best of our knowledge, not yet been studied in the directional change framework.

Directional change (DC) methods represent price changes of an asset only when the change in the asset's price compared to the last recorded price exceeds a selected threshold [10], [11]. The DC representation of an asset price complements the standard, fixed time interval, asset price data in terms of understanding the data features. Relationships between different types of events within the DC representations of an asset price have been studied in detail. A set of scaling laws is shown to provide accurate estimates of expected price changes and expected duration of price changes [12]. DC representations and associated scaling laws have been used to obtain trading strategies for individual assets [13]–[16]. Recently, the DC framework was extended to represent multiple assets for the purpose of DC-based dynamic portfolio optimization [17].

In this paper, we study a simple robust DC-based portfolio optimization (RDC) method, for returns maximization. The proposed method joins the price signals obtained from DC representations of multiple assets and robust portfolio optimization. This robust portfolio optimization objective function maximizes the portfolio returns while taking into account return uncertainty. The maximization for portfolio returns is in line with the empirical scaling laws of DC representations of asset prices, which have implications for expected asset prices and returns, specifically their relation to the selected threshold level, expected continuation of an upward or downward price movement, and the expected return until the opposite DC point [12]. The proposed method does not explicitly incorporate variance or covariance uncertainty in assets and it does not explicitly aim for risk diversification.

We empirically study the effect of the robust DC-based portfolio optimization method. We apply the proposed method to a portfolio consisting of a set of diversified assets: 29 Exchange

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Traded Funds (ETF) for the period between 2 January 2018 and 30 December 2021. We compare the obtained portfolio results with benchmarks and show that the proposed method performs comparably to several benchmarks. The proposed method defines a simple robust optimization rule which maximizes returns. Despite its simplicity, we find promising results where metrics such as maximum drawdown improve compared to benchmarks in the long investment period we analyze.

II. Portfolio Optimization and Directional Change Representations of Multiple Time Series

A. Directional Changes and Intrinsic Time Series Representation

Directional Change (DC) models aim to create intrinsic time series that summarize upward and downward market price movements of a financial instrument [14]. The intrinsic time series consists of time periods where the price change exceeds a given pre-defined threshold value θ. A downturn DC event is defined when the difference between the current price p_t and the last high price p_h is lower than θ :

$$p_t \leq p_h(1-\theta). \tag{1}$$

Conversely, an upturn DC event is defined when the difference between the current market price p_t and the last low price p_l is higher than a fixed threshold θ:

$$p_t \geq p_l(1+\theta). \tag{2}$$

A downward (upward) DC event is followed by a downward (upward) overshoot (OS) until an opposite, upward, or downward DC event occurs. The combination of these DC and OS events consists of the intrinsic time series for a given θ [12], [13]. It is shown that the ratio of the length of OS to DC events is a reliable signal for market movements particularly compared to observed prices [14].

The series of DC and OS events are defined as follows: Let $p_{n,t}$ denote the price level of asset $n = 1, \dots, N$ at observed time intervals $t = 1, \dots, T$. Furthermore, let $r_{n,t} = 100 \times \ln(p_{n,t} - p_{n,t-1})$ denote the percentage returns of asset n at time t. Define $\mathrm{DC}_{n,0}$ as the initial direction of asset n at intrinsic time $k = 0$. For $k = 1, \dots, K_n$ directional change points with $K_n < T$, each DC and its sign is calculated iteratively based on $\mathrm{DC}_{n,k-1}$.

$$\mathrm{DC}_{n,k} = \underset{t \geq \mathrm{DC}_{n,k-1}}{\arg\min} \left(p_{n,t} \geq p_{n,\mathrm{DC}_{n,k-1}}(1+\theta)\right) \tag{3}$$

where $p_{n,h} = p_{\mathrm{DC}_{n,k}}$ is the last 'high price' in the market, and 'event' $\mathrm{EDC}_{n,k} = \mathrm{UDC}$ is the sign of the DC. If $\mathrm{DC}_{n,k-1}$ is an upward DC, the next directional change is a DDC:

$$\mathrm{DC}_{n,k} = \underset{t \geq \mathrm{DC}_{n,k-1}}{\arg\min} \left(p_{n,t} \leq p_{n,\mathrm{DC}_{n,k-1}}(1-\theta)\right) \tag{4}$$

where $p_{n,l} = p_{n,\mathrm{DC}_{n,k}}$ is the last 'low price' in the market, $\mathrm{EDC}_{n,k} = \mathrm{DDC}$ and $\mathrm{DC}_{n,0}$ is initialized.

In this paper, we use a set of summary metrics for each asset n for portfolio optimization. These metrics are the average time of a DC_n (TDC_n), the average time of an OS_n (TOS_n) after a confirmed directional change, and the average ratio of OS event length over the average ratio of DC event length (TOS_n/TDC_n):

$$\mathrm{TDC}_n = \frac{1}{K_n} \sum_{k=1}^{K_n} \left(\mathrm{DC}_{n,k} - \mathrm{DC}_{n,k-1}\right), \tag{5}$$

$$\mathrm{TOS}_n = \frac{1}{K_n} \sum_{k=1}^{K_n} \left(\mathrm{OS}_{n,k} - \mathrm{DC}_{n,k}\right), \tag{6}$$

$$\mathrm{TOS}_n/\mathrm{TDC}_n = \frac{1}{K_n} \sum_{k=1}^{K_n} \frac{\mathrm{OS}_{n,k} - \mathrm{DC}_{n,k}}{\mathrm{DC}_{n,k} - \mathrm{DC}_{n,k-1}}. \tag{7}$$

B. Portfolio Optimization using joint DCs

Recently, the DC representation approach was extended to define a joint intrinsic time series for multiple assets and portfolio optimization [17]. This joint intrinsic time series representation of a portfolio DC point, PDC_k, is defined when one or more assets have a confirmed DC:

$$\mathrm{PDC}_k = \underset{t \geq \mathrm{PDC}_{k-1}}{\arg\min}\ \exists n \text{ s.t. } \mathrm{DC}_{n,k} = t\,, \tag{8}$$

where $k = 1, \dots, K$ and PDC_0 is initialized at time 0, similar to the DC approach for individual assets.

From the joint DCs, joint OS events are obtained at each point k:

$$\mathrm{OS}_{n,k} = \begin{cases} \mathrm{OS}_{n,k} & \text{if } \mathrm{PDC}_k \in \mathrm{DC}_n, \\ \mathrm{OS}_{n,k-1} & \text{otherwise}, \end{cases} \tag{9}$$

where $\mathrm{DC}_n = \{\mathrm{DC}_{n,1}, \dots, \mathrm{DC}_{n,K_n}\}$ is the set of DC points for asset n. The remaining assets are assumed to follow the direction they had at time $k-1$.

Based on the joint DC representations, the portfolio weights are determined using the relative expected return and risk properties from historical OS events:

$$\hat{r}_{n,\mathrm{UDC}} = \frac{\left(p_{\mathrm{OS}_{n,k}} - p_{\mathrm{OS}_{n,k-1}}\right) I\left(\mathrm{EDC}_{n,k} = \mathrm{UDC}\right)}{\sum_{k=2}^{K} I\left(\mathrm{EDC}_{n,k} = \mathrm{UDC}\right)}, \tag{10}$$

$$\hat{r}_{n,\mathrm{DDC}} = \frac{\left(p_{\mathrm{OS}_{n,k}} - p_{\mathrm{OS}_{n,k-1}}\right) I\left(\mathrm{EDC}_{n,k} = \mathrm{DDC}\right)}{\sum_{k=2}^{K} I\left(\mathrm{EDC}_{n,k} = \mathrm{DDC}\right)}, \tag{11}$$

$$\hat{s}_n^2 = \frac{1}{K-1} \sum_{k=2}^{K} \left(p_{\mathrm{OS}_{n,k}} - p_{\mathrm{OS}_{n,k-1}}\right)^2, \tag{12}$$

where $I[.]$ is an indicator function which takes the value of 1 if its argument is true, and the value of 0 otherwise. Furthermore, $p_{\mathrm{OS}_{n,k}}$ is defined as the price at the end of the OS period following the kth directional change for asset n.

An example of a joint DC representation for two artificially generated asset prices is provided in Fig. 1, where the prices of two assets are shown together with the corresponding DC and OS events. In this figure, the vertical lines represent the confirmed DC points for the joint intrinsic time series representation for the two assets, where at each DC point at least one asset has a confirmed directional change as defined in (8).

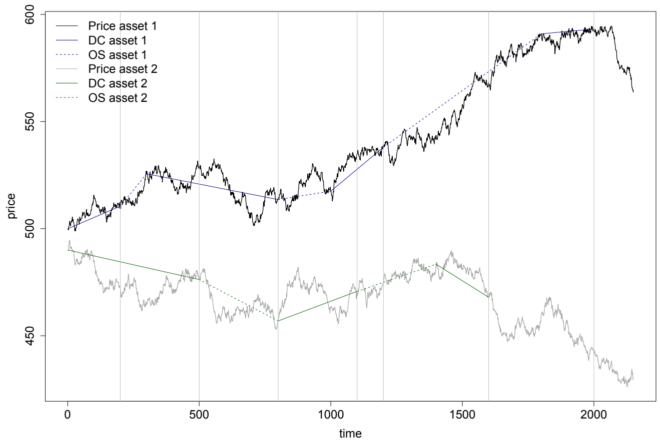

Fig. 1. DC representations of two artificially generated asset prices.

III. Robust Portfolio Optimization for joint DC representations

We present the proposed DC-based robust portfolio optimization method. In the literature, robust portfolio optimization is developed for the standard time series representations of multiple assets [7], [8]. It is considered as a robust counterpart of the mean-variance framework [18] in order to mitigate the sensitivity of a portfolio the input parameters such as returns where the method optimizes the worst case by defining uncertainty sets of uncertain parameters [19].

In robust portfolio optimization, the set of uncertain parameters can be defined in different ways, such as robustness to returns and variance or Sharpe ratio uncertainty robustness to the covariance matrix or the inverse of the covariance matrix of returns [20]. Despite a large number of robust portfolio proposals, the literature is not conclusive on which robust portfolio optimization methods empirically or theoretically lead to more stable portfolio results or better out-of-sample performance [21].

In this paper, we define robust portfolio optimization to maximize portfolio returns. This objective tries to make use of the DC representations of assets and their empirical scaling laws which relate to the price changes and expected duration of price changes [12]. The optimization function for RDC is defined, based on [8], as follows:

$$\max \min_{z \in Z} \sum_{n=1}^{N} (\hat{r}_n + \hat{s}_n z_n)\omega_n \tag{13}$$

$$\text{s.t.} \sum_{i=1}^{N} \omega_n = 1, \tag{14}$$

$$\omega_n \geq 0, n = 1, \ldots, N, \tag{15}$$

where $\hat{r}_n$ and $\hat{s}_n$ is the expected return and the risk metrics of asset n, obtained from the DC representations of the assets, respectively. In this notation, ω_n are the weights of each asset in the portfolio and they determine the portfolio allocation. Dynamic portfolio allocation corresponds to optimizing these weights at each portfolio rebalancing point.

The optimization in (13) is based on [8], where we define the expected return and risk metrics according to the historical OS events for upward and downward movements in equations (10), (11) and (12). In this formulation, set Z determines the ‘robust model’ which allows for uncertainty in expected returns from historical OS events:

$$Z = \left\{ \{z_1, \ldots, z_N\} : 0 \leq z_n \leq 1, \forall n; \sum_{n=1}^{N} z_n \leq \Gamma \right\} \tag{16}$$

where Γ relates to the conservatism of the model [8]. It is expected that an increase in Γ, the protection parameter, decreases the extreme returns such as maximum drawdown. The main motivation for this step is the observation of the experimental literature that shows that investors perceive the loss potential as more important than volatility, see, e.g., [22].

Within the DC approach, we obtain the expected risk and return properties of each asset using (10)-(12) for each time period that the joint DC representation indicates portfolio rebalancing. The portfolio is rebalanced when the expected portfolio return is larger than zero, i.e.

$$\sum_{n=1}^{N} \hat{r}_n \omega_n > 0. \tag{17}$$

This additional step to rebalance the portfolio only for positive expected returns indicates that the number of portfolio rebalancing points can be smaller than the number of confirmed DC points in the joint intrinsic time series representation.

IV. Application to Daily ETF Data

We apply the RDC optimization to daily ETF data and present the results of the obtained portfolio compared to the benchmarks. Note that the methodology discussed in Section II uses only prices and expected returns of the assets. It does not explicitly account for risk diversification since it does not incorporate measures of co-movements between assets. As such, we expect the method to work best when using assets that already are somewhat diversified. ETF data has such diversification properties since each ETF typically holds a basket of assets. We use ETFs to apply the methodology discussed in Section II since Equation (13) uses only the expected returns of the assets. As such, we expect the method to work best when using assets that already are somewhat diversified.

The data are obtained from the mutual fund database of the Center for Research in Security Prices (CRSP) as the daily total returns for the ETFs between January 2005 - December 2021. We chose the largest equity ETFs in terms of average net asset values within each style category provided by CRSP. Finally, the risk-free rate is taken from the Fama-French factor database of Kenneth R. French, accessed through Wharton Research Data Services. In the DC representation, the threshold parameter determines how large a change is considered significant, and price changes that are smaller than a threshold are ignored [14]. Due to this property, the DC representation can be seen as a noise-filtering method. We therefore do not filter the data prior to DC analysis.

The application of the RDC method depends on the selection of two parameters, the DC threshold θ and the uncertainty budget Γ. Intuitively, the number of DC points and portfolio rebalancing points decrease with θ while the obtained portfolio returns are more robust to return uncertainty with increasing Γ. These properties, however, do not translate directly to the properties, such as the Sharpe ratio, of out-of-sample realized returns. We, therefore, report the application results for several values of θ and Γ. Following [8] we report the results for a low, medium, and high level of Γ where a higher Γ indicates higher robustness to price uncertainty. We apply the DC algorithm with, $\theta = 0.04$ (RDC 004), $\theta = 0.05$ (RDC 005), and $\theta = 0.06$ (RDC 006) and robust portfolio optimization parameter $\Gamma = 5$, $\Gamma = 20$ and $\Gamma = 50$ and denote the results e.g. as RDC 004;5 for $\theta = 0.04$ and $\Gamma = 5$.

A. RDC portfolio investment results

We include a comparison to the naive 1/N strategy as this is a well-diversified strategy and a comparison to the minimum variance strategy to see the relative performance of the RDC method compared to a strategy that takes a different approach to model portfolio risk. Note that DCs and the portfolio rebalancing points we define in Section II-B do not coincide with fixed time intervals. DCs happen at non-fixed periods of time and our rebalancing points are not fixed. We, therefore, initialize the RDC strategy by investing according to the 1/N strategy until the first rebalancing point. This ensures that all strategies start at the same time point.

TABLE I
ROLLING WINDOW PORTFOLIO RESULTS

	Exc. Ret.	Std. Dev.	Sharpe Ratio	$\hat{z}_{\mathrm{JK}}$	M.Draw.
S&P 500	0.087	0.251	0.348	-1.020	-0.651
1/N	0.107	0.193	0.554	-0.545	-0.579
Min. Var.	0.103	0.142	0.726	0.537	-0.465
RDC 004;5	0.107	0.193	0.553	-	-0.579
RDC 004;20	0.113	0.173	0.652	-	-0.393
RDC 004;50	0.113	0.173	0.652	-	-0.393
RDC 005;5	0.117	0.213	0.548	-	-0.622
RDC 005;20	0.062	0.237	0.263	-	-0.647
RDC 005;50	0.062	0.237	0.263	-	-0.647
RDC 006;5	0.132	0.216	0.610	-	-0.592
RDC 006;20	0.107	0.193	0.553	-	-0.579
RDC 006;50	0.107	0.193	0.553	-	-0.579

Table I presents the standard portfolio evaluation measures, the mean, variance and Sharpe Ratio of these realized returns, which are out of sample observations during which the portfolio was held. We also report the statistical significance of the differences in Sharpe Ratios [23] in the second to last column which shows the realized values for a test that is asymptotically standard normal. Finally, we show the maximum drawdown experienced by the portfolio strategies within the period in the last column of the table.

The statistics of most interest are the Sharpe ratio and the maximum drawdown, as they relate the return to the risk of the portfolio strategy. Concerning the first measure, we see that the RDC with $\theta = 0.04$ and $\Gamma = 20$ strategy yields the second largest Sharpe ratio. However, this result is statistically indistinguishable from the simple buy-and-hold and the naive diversification strategies, both resulting in lower realized Sharpe ratios. In addition, the minimum variance strategy, which yields the largest realized Sharpe ratio, is also statistically indistinguishable from the RDC 004;20 results.

Since the robust optimization technique has the goal to limit large losses we now turn to the maximum drawdown results. We see that the best-performing RDC strategy in terms of the Sharpe ratio is able to achieve this in combination with the smallest absolute value for the maximum drawdown. Note that the drawdown of -39.3% is the lowest realized drawdown of all strategies. It is also interesting to see that, although achieving the lowest value for the standard deviation, the minimum variance strategy has a drawdown of -46.5%.

Concerning the calibration for the parameters θ and Γ we have the following observations. The parameter θ controls the threshold for a directional change and we see that for a small value, the robust component of the optimization matters, i.e., large values of Γ improve the Sharpe ratio and the values for the maximum drawdown. In contrast, for large values of θ, i.e., a larger threshold for the directional change, the value for Γ is less important, if anything, larger values have a detrimental effect on the performance.

The results in Table I indicate two conclusions for these calibration parameters. First, the results for $\Gamma = 20$ and $\Gamma = 50$ are indistinguishable across all values for θ. This indicates pushing this parameter beyond a certain value does not lead to performance differences. Second, there is no clear pattern in the results concerning Sharpe ratios. Despite this, we observe that an increase in θ leads to a loss in performance with respect to the maximum drawdown. The same conclusion holds for the realized standard deviations.

We conclude from this analysis that it is possible to construct a portfolio strategy using RDC that is able to generate a performance that is on par with the benchmark strategies in terms of Sharpe ratios and is able to generate a lower maximum drawdown. This performance is found for parameter values $\theta = 0.04$ and $\Gamma = 20$.

V. CONCLUSION

We combine DC methods with robust optimization tools to set up a portfolio optimization strategy. The algorithm is applied to an asset menu consisting of 29 exchange-traded funds. We find that the algorithm does not produce significantly different from the benchmark methods in terms of Sharpe ratios but can improve on the maximum drawdowns realized in the investment period under consideration. Since the robust part of the optimization algorithm has the goal to limit the downside risk in the portfolio strategy we view this as a promising result.

As future work, we want to extend our methodology to other robust portfolio optimization rules that incorporate uncertainty in estimated variances and covariances of a portfolio [20], but

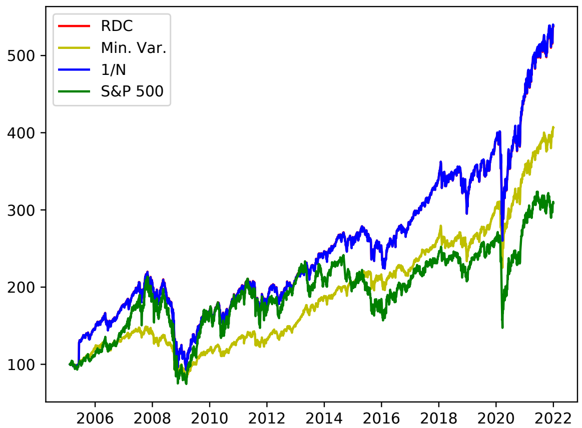

Fig. 2. Rolling window portfolio results with $\theta = 0.04$ and $\Gamma = 5$.

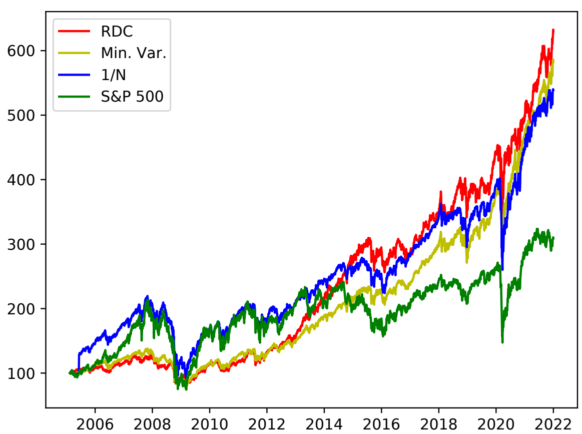

Fig. 3. Rolling window portfolio results with $\theta = 0.04$ and $\Gamma = 20$.

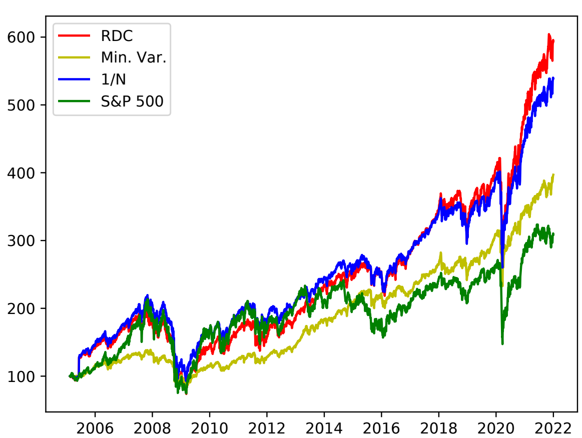

Fig. 4. Rolling window portfolio results with $\theta = 0.05$ and $\Gamma = 5$.

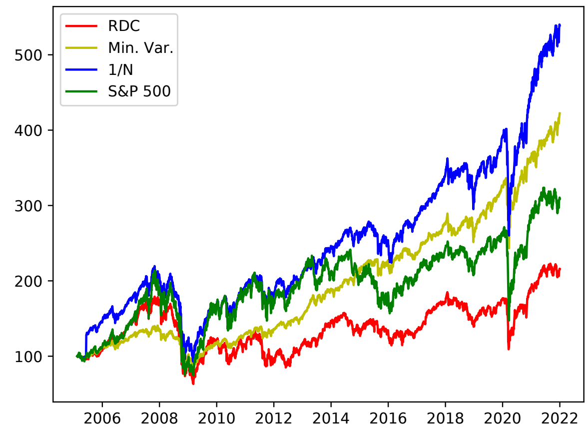

Fig. 5. Rolling window portfolio results with $\theta = 0.05$ and $\Gamma = 20$.

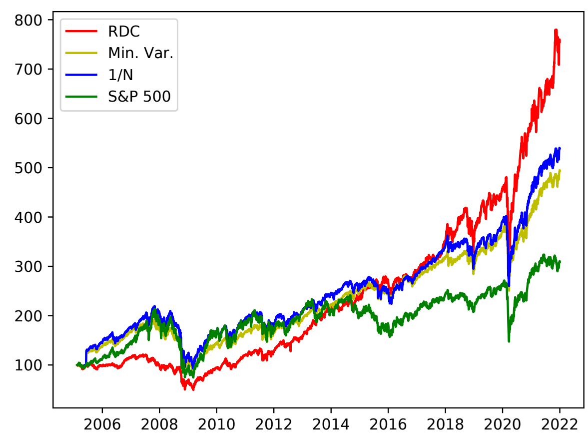

Fig. 6. Rolling window portfolio results with $\theta = 0.06$ and $\Gamma = 5$.

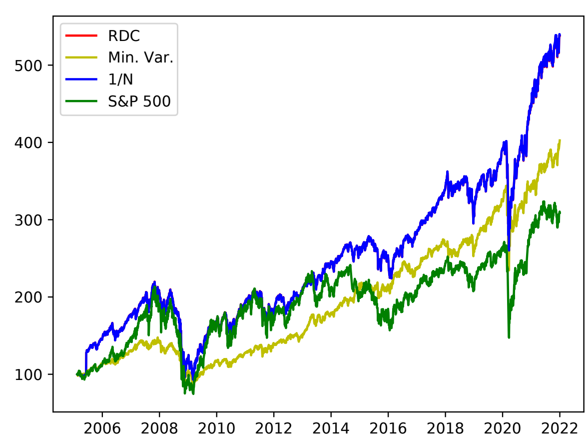

Fig. 7. Rolling window portfolio results with $\theta = 0.06$ and $\Gamma = 20$.

such an extension requires the study of DC representations of variance and covariance matrices.

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