Contraction Based Synchronisation of Complex Delayed Dynamical Network with Power Leader

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Abstract— **This paper investigates the leader-follower synchronization problem of delayed complex dynamical network. The proposed methodology analytically derives the synchronisation condition for complex delayed dynamical network with a power leader such that all follower systems (delayed or non-delayed) exponentially synchronize their states to power leader through local interactions. For developing the explicit conditions for synchronisation, partial contraction theory along with properties of coupling matrices are exploited. Simulation results are used to demonstrate the effectiveness of the proposed control algorithm.**

I. INTRODUCTION

Complex network refers to network of interconnected elements or nodes, which is a fundamental concept in the field of network science and has been applied to diverse array of science and engineering disciplines [1], [2]. The application of complex network analysis to leader-follower networks provides a powerful tool for understanding and modeling the relationships between the leader and followers. Designing and analyzing control strategies for achieving and maintaining synchronization in such networks have been prominently explored over the years. One of the key applications of complex network analysis in the context of leader-follower networks is the study of synchronization. The such synchronization in leader-follower networks can be beneficial in variety of applications in real-world including robotics, power system communication networks, social networks, biological systems and so on [3], [4]. These are just a few examples of the practical applications of synchronization of leader-follower networks. The objective of synchronization in leader-follower networks is to make the followers behavior match the behavior of the leader. Studies have shown that synchronization in leader-follower networks can be achieved through various methods, including linear and nonlinear feedback control, adaptive control, and distributed consensus algorithms etc. [5], [6]. The choice of method depends on the dynamics and communication architecture of the leader and follower systems. Overall, the literature on synchronization in leader-follower networks highlights the importance of considering the interplay between network structure, control strategies, and system dynamics for achieving and maintaining synchronization in complex networked systems. The synchronization of leader-follower network has the potential to provide solutions and insights into a wide range of real-world problems, and continues to be an active area of research and development.

In the literature, numerous studies have made significant contributions to the leader-follower synchronization problem [7], [8], [9]. Additionally, researchers have considered the impact of uncertainties and external disturbances on synchronization of systems in leader-follower networks [10], [11]. The leader-follower synchronisation aspect has been investigated in various environments, including those in which all controller agents have perfect knowledge of the leader's trajectory [12], each agent controller has an internal model that captures the dynamics of the leader's desired trajectory, the leader is accessible to a finite number of followers [13], and when there are multiple leaders [14]. There is a notable gap in the literature regarding the synchronization issues specific to power leader networks. In commonly used approaches for such synchronisation, generally the leader node is connected to all the followers or multiple leader nodes are utilized, Here, network involves the leader node which is selectively connected to a few follower nodes, while the other followers engage in time-delayed communication. This is a unique representation of many real-world systems, and it is essential to analyse and understand the synchronization problems of these networks. The majority of results available in literature utilize Lyapunov stability theory in the background for achieving the goal. Recently, contraction theory as a design tool has also gained popularity for addressing stability & synchronisation of systems. Contraction theory is utilized to study convergence behaviour of the neighbouring trajectories within a nonlinear system. Contraction theory and its application have been applied to a broad range of nonlinear systems, including dynamic systems, differential equations, and scientific and engineering disciplines [15], [16], [17]. In recent literature, researchers have been actively addressing the synchronization problem in nonlinearly coupled networks through the utilization of various discontinuous control strategies. These strategies include, but are not limited to pinning control, intermittent control, and event-triggered control (ETC) strategies [18], [19]. These discontinuous control approaches offer novel and effective ways to achieve synchronization in complex networks by exploiting intermittent interventions and event-based triggering mechanisms. All these methods require knowledge of the network topology and are sensitive to parameter uncertainties. The choice of method depends on the system and application. Here, in present works, proposed contraction based control strategy present a notable advantage in its wide applicability to a range of identical or non-identical nonlinear time-varying systems. It can effectively handle complex coupling configurations and multiple time delays, thus, making it suitable for various practical applications.

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This article provides synchronisation conditions for complex delayed dynamical networks with a power leader, where not all followers are directly connected to power leader. Overall, the primary aspects of the present work are highlighted as:

- Analytical synchronization criterion for delayed complex dynamical networks with one power leader is presented.
- Using partial contraction analysis and explicit properties of coupling matrices along with unforced system dynamics, the conditions for exponential synchronisation of complex network with power leader are derived.
- Comprehensive numerical simulations are performed on a distinct delayed leader-follower network model to validate the efficacy of theoretical results. The proposed methodology may be extended to other synchronization problems or network topologies as well.

The outline of this paper is summarized as follows. In section II, we briefly describe the concept of contraction analysis. In section III, we derive Leader-follower synchronization control strategy for delayed dynamical network of diffusively coupled nonlinear systems using contraction theory. In section IV, numerical simulation results using MATLAB are presented to verify the proposed strategy. In section V, some concluding remarks are drawn regarding proposed contribution.

II. Contraction Theory

Contraction theory is a mathematical framework for the analysis and control of nonlinear systems. The main idea behind contraction theory is that nonlinear systems can be modeled as metric spaces, and that the evolution of these systems can be characterized by the contraction of distances between pairs of states. The theory states that if the Jacobian matrix of a nonlinear system satisfies certain contraction condition, then the system is guaranteed to converge to a unique equilibrium state, regardless of the initial conditions. This convergence is exponentially fast and robust to small disturbances [20].

A. Contraction Analysis

Let us consider a nonlinear system described by the differential equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

Here, $\mathbf{x}$ represents an n-dimensional vector of states, a nonlinear vector function $\mathbf{f}$ defines the system dynamics. The first variation of (1) is defined as

$$\delta\dot{\mathbf{x}} = \delta\mathbf{f} = \frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}}\delta\mathbf{x} \tag{2}$$

By using the derivative as above, we can write,

$$\begin{aligned}\frac{d}{dt}(\delta\mathbf{x}^T\delta\mathbf{x}) = 2(\delta\mathbf{x}^T\delta\dot{\mathbf{x}}) = 2\delta\mathbf{x}^T\frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}}\delta\mathbf{x} \\ \leq 2\lambda_m(\mathbf{x},t)(\delta\mathbf{x}^T\delta\mathbf{x})\end{aligned} \tag{3}$$

Here, $(\delta\mathbf{x}^T\delta\mathbf{x})$ defines the squared distance between two neighbouring trajectories. $\lambda_m(\mathbf{x},t)$ defines the maximum eigenvalue of the symmetric part of the associated Jacobian matrix $\frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}}$. If $\frac{\partial \mathbf{f}(\mathbf{x},t)}{\partial \mathbf{x}}$ is proved to be uniformly negative definite (UND), then any infinitesimal length $\|\delta\mathbf{x}\|$ will exponentially approach to zero. Consequently, all solution trajectories converge exponentially to zero, independent of their initial conditions. Further developments in contraction theory encompass the concept of partial contraction analysis, which includes convergence to specific behaviors or properties instead of individual trajectories. Results related to partial contraction are detailed in [21].

Lemma 1: Consider a non-linear system of the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{x}, t) \tag{4}$$

corresponding auxiliary system is defined as

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{x}, t) \tag{5}$$

If this system is contracting with respect to $\mathbf{y}$ and solution of the $\mathbf{y}$-system coverage to particular behaviour or to smooth specific property, then $\mathbf{x}$-system trajectories also converge to the independent property, exponentially. The original $\mathbf{x}$-system is said to be partial contracting.

III. Main Results

Consider a complex and delayed dynamical network with a power leader and N-follower nodes as follows:

$$\begin{aligned}\dot{\mathbf{x}}_0 =& \mathbf{f}(\mathbf{x}_0, t) \\ \dot{\mathbf{x}}_i =& \mathbf{f}(\mathbf{x}_i, t) + \sum_{j\in\mathscr{N}_i} k_1(\mathbf{x}_j - \mathbf{x}_i) + \gamma_i k_0(\mathbf{x}_0 - \mathbf{x}_i) \\ &+ \sum_{j\in\mathscr{H}_i} k_2(\mathbf{x}_j(t-\tau_{ji}) - \mathbf{x}_i)\end{aligned} \tag{6}$$

Here, every node is assumed to have n-dimension. The state of the leader is denoted as $\mathbf{x}_0$ and N non-identical non-linear dynamical systems are represented as $\mathbf{x_i}$ for $i = 1, 2, \ldots, N$. The states of each of the followers is represented by $\mathbf{x_i} \in \mathfrak{R}^n$ with vector flow function as $\mathbf{f}$. Further, $\mathbf{f}(\mathbf{x_i}, t)$: $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ represents the dynamics of uncoupled nonlinear systems. The relationships among the followers are characterized by $\mathscr{N}_i$. The coupling force between the leader and its corresponding followers is characterised by k_0. γ_i is indicating the direct connection from leader to the followers, which is either 0 or 1. Here, k_1 and k_2 are the gains associated with the non-delay and delay coupling, respectively. τ_{ji} is the coupled delay and assumed to be bounded by a known constant i.e., $0 < \tau_{ji} < \tau$.

Theorem 1: The complex delay dynamical network given in (6) with power leader can achieve leader-follower synchronization if the following conditions is satisfied:

$$[\lambda_{max}(I^n_{\mathscr{J}F}) - \lambda_{max}(\mathscr{L}_{k_1} + \gamma_i k_0 \mathbf{I}_n)] < 0$$

$$\frac{1}{2}k_2^{-1} > 0$$

where $(\lambda_{max}(I^n_{\mathscr{J}\mathscr{F}})$ is the maximum eigenvalue of uncoupled system dynamics. $\lambda_{max}(\mathscr{L}_{k_1} + \gamma_i k_0 \mathbf{I}_n)$ is the maximum eigenvalue of combination of sum of Laplacian of the network with the identity matrix of gain $\gamma_i k_{0i}$. The conditions stated above guarantee exponential convergence of all the states of follower systems to the states of the leader system, irrespective of their initial values, provided that $\lambda_{max}(I^n_{\mathscr{J}\mathscr{F}})$ is upper bounded.

Proof: For the system in (6), One can construct a simplified

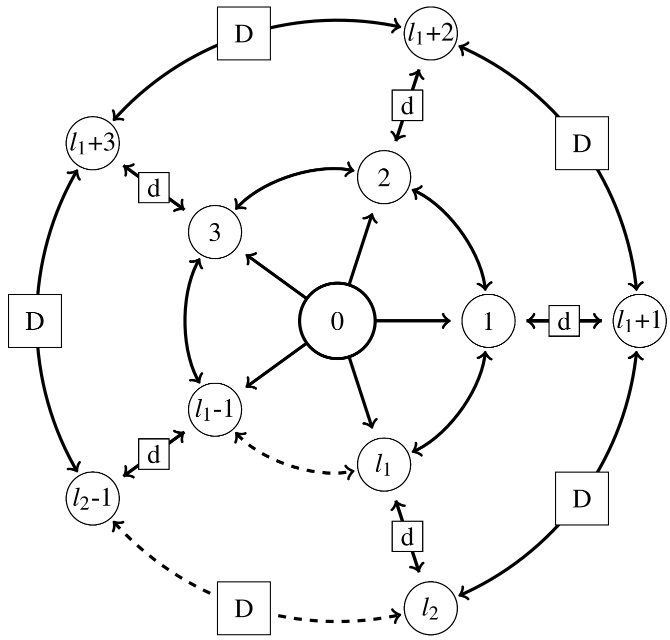

Fig. 1. Complex delay dynamical network with power leader

auxiliary system as

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i,t) + \sum_{j\in\mathscr{N}_i} k_1(\mathbf{y}_j - \mathbf{y}_i) + \gamma_i k_0(\mathbf{x}_0 - \mathbf{y}_i) + \sum_{j\in\mathscr{H}_i} k_2(\mathbf{y}_j(t-\tau_{ji}) - \mathbf{y}_i) \tag{7}$$

The equation (7) can be transformed to

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i,t) + \sum_{j\in\mathscr{N}_i} k_1(\mathbf{y}_j - \mathbf{y}_i) + \gamma_i k_0(\mathbf{x}_0 - \mathbf{y}_i) + \sum_{j\in\mathscr{H}_i} \zeta_{ji} \tag{8}$$

where ζ_{ji} and corresponding ζ_{ij} are defined as
$\mathbf{u}_{ji} = k_2\mathbf{y}_i + \zeta_{ji}$; $\mathbf{v}_{ij} = k_2\mathbf{y}_i$; $\mathbf{u}_{ji} = \mathbf{v}_{ji}(t-\tau_{ji})$
$\mathbf{u}_{ij} = k_2\mathbf{y}_j + \zeta_{ij}$; $\mathbf{v}_{ji} = k_2\mathbf{y}_j$; $\mathbf{u}_{ij} = \mathbf{v}_{ij}(t-\tau_{ij})$
The above mapping of variables is similar to the case as given in [9]. Here, τ_{ji} is defined as communication time delay between i_{th} and j_{th} system. The dynamics in (8) can be expressed in terms of differential framework as follows

$$\delta\dot{\mathbf{y}}_i = \frac{\partial \mathbf{f}(\mathbf{y}_i,t)}{\partial \mathbf{y}_i}\delta\mathbf{y}_i + \sum_{j\in\mathscr{N}_i} k_1(\delta\mathbf{y}_j - \delta\mathbf{y}_i) - \gamma_i k_0(\delta\mathbf{y}_i) + \sum_{j\in\mathscr{H}_i} \delta\zeta_{ji} \tag{9}$$

Consider the differential Lyapunov quadratic function, which is expressed as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}(\delta\mathbf{y}_i^T\delta\mathbf{y}_i) + \frac{1}{2k_2}\sum_{(i,j)\in\mathscr{N}} \mathbf{v}_{ij} \tag{10}$$

where $\mathscr{N} = \cup_{i=1}^{N}\mathscr{H}_i$ represents the set of all active links. $\mathbf{v}_{ij}$ is defined as

$$\mathbf{v}_{ij} = \int_{t-\tau_{ij}}^{t} \delta\mathbf{v}_{ij}^T\delta\mathbf{v}_{ij}ds + \int_{t-\tau_{ji}}^{t} \delta\mathbf{v}_{ji}^T\delta\mathbf{v}_{ji}ds$$

$$\mathbf{v}_{ij} = \int_0^t \left\{\delta\mathbf{v}_{ij}^T\delta\mathbf{v}_{ij} - \delta\mathbf{v}_{ij}^T(t-\tau_{ij})\delta\mathbf{v}_{ij}(t-\tau_{ij})\right\}ds + \int_0^t \left\{\delta\mathbf{v}_{ji}^T\delta\mathbf{v}_{ji} - \delta\mathbf{v}_{ji}^T(t-\tau_{ji})\delta\mathbf{v}_{ji}(t-\tau_{ji})\right\}ds$$

By using definitions of $\mathbf{u}_{ij}$ and $\mathbf{u}_{ji}$, the above expression simplifies to

$$\mathbf{v}_{ij} = \int_0^t \left\{\delta\mathbf{v}_{ij}^T\delta\mathbf{v}_{ij} - \delta\mathbf{u}_{ij}^T\delta\mathbf{u}_{ij}\right\}ds + \int_0^t \left\{\delta\mathbf{v}_{ji}^T\delta\mathbf{v}_{ji} - \delta\mathbf{u}_{ji}^T\delta\mathbf{u}_{ji}\right\}ds$$

Taking the time derivative of V(t) in (10), one can write,

$$\dot{V}(t) = \sum_{i=1}^{N}(\delta\mathbf{y}_i^T\delta\dot{\mathbf{y}}_i) + \frac{1}{2k_2}\sum_{(i,j)\in\mathscr{N}}\dot{\mathbf{v}}_{ij}$$

Using (9), we can write,

$$\dot{V}(t) = \sum_{i=1}^{N}\delta\mathbf{y}_i^T\Big(\mathscr{J}\mathscr{F}\delta\mathbf{y}_i + \sum_{j\in\mathscr{N}_i}k_1(\delta\mathbf{y}_j - \delta\mathbf{y}_i) - \gamma_i k_0\delta\mathbf{y}_i + \sum_{j\in\mathscr{H}_i}\delta\zeta_{ji}\Big) + \frac{1}{2k_2}\sum_{(i,j)\in\mathscr{N}}\Big\{\delta\mathbf{v}_{ij}^T\delta\mathbf{v}_{ij} - \delta\mathbf{u}_{ij}^T\delta\mathbf{u}_{ij} + \delta\mathbf{v}_{ji}^T\delta\mathbf{v}_{ji} - \delta\mathbf{u}_{ji}^T\delta\mathbf{u}_{ji}\Big\}$$

$$\dot{V}(t) = \sum_{i=1}^{N}\delta\mathbf{y}_i^T\Big(\mathscr{J}\mathscr{F}\delta\mathbf{y}_i + \sum_{j\in\mathscr{N}_i}k_1(\delta\mathbf{y}_j - \delta\mathbf{y}_i) - \gamma_i k_0\delta\mathbf{y}_i + \sum_{j\in\mathscr{H}_i}\delta\zeta_{ji}\Big) + \frac{1}{2k_2}\sum_{(i,j)\in\mathscr{N}}\Big\{\delta\mathbf{v}_{ij}^T\delta\mathbf{v}_{ij} - (\delta\mathbf{v}_{ji}^T + \delta\zeta_{ij}^T)(\delta\mathbf{v}_{ji} + \delta\zeta_{ij}) + \delta\mathbf{v}_{ji}^T\delta\mathbf{v}_{ji} - (\delta\mathbf{v}_{ij}^T + \delta\zeta_{ji}^T)(\delta\mathbf{v}_{ij} + \delta\zeta_{ji})\Big\}$$

$$\dot{V}(t) = \delta\mathbf{y}^T\Big(\mathscr{J}\mathscr{F}\otimes\mathbf{I}_n - \mathscr{L}_{k_1}\otimes\mathbf{I}_n - \gamma k_0\otimes\mathbf{I}_n\Big)\delta\mathbf{y} + \sum_{i=1}^{N}\delta\mathbf{y}^T\sum_{j\in\mathscr{H}_i}\delta\zeta_{ji} + \frac{1}{2k_2}\sum_{(i,j)\in\mathscr{N}}\Big\{\delta\mathbf{v}_{ij}^T\delta\mathbf{v}_{ij} - \Big(\delta\mathbf{v}_{ji}^T\delta\mathbf{v}_{ji} + 2k_2\delta\mathbf{y}_i^T\delta\zeta_{ij} + \delta\zeta_{ij}^T\delta\zeta_{ij}\Big) + \delta\mathbf{v}_{ji}^T\delta\mathbf{v}_{ji} - \Big(\delta\mathbf{v}_{ij}^T\delta\mathbf{v}_{ij} + 2k_2\delta\mathbf{y}_j^T\delta\zeta_{ji} + \delta\zeta_{ji}^T\delta\zeta_{ji}\Big)\Big\}$$

It further gets simplified to following:

$$\dot{V}(t) = \delta\mathbf{y}^T\Big(\mathscr{J}\mathscr{F}\otimes\mathbf{I}_n - \mathscr{L}_{k_1}\otimes\mathbf{I}_n - \gamma k_0\otimes\mathbf{I}_n\Big)\delta\mathbf{y} - \frac{1}{2k_2}\sum_{(i,j)\in\mathscr{N}}\Big(\delta\zeta_{ij}^T\delta\zeta_{ij} + \delta\zeta_{ji}^T\delta\zeta_{ji}\Big)$$

Finally, the derivative $\dot{V}(t)$ reduces to

$$\dot{V}(t) \leq \delta\mathbf{y}^T\Big(\lambda_{max}(\mathscr{J}\mathscr{F}) - \lambda_{max}(\mathscr{L}_{k_1} + \gamma k_0\mathbf{I}_n)\Big)\delta\mathbf{y} - \frac{1}{2}\sum_{(i,j)\in\mathscr{N}}\Big(\delta\zeta_{ij}^T k_2^{-1}\delta\zeta_{ij} + \delta\zeta_{ji}^T k_2^{-1}\delta\zeta_{ji}\Big)$$

Note that the gains $k_1 > 0$; $k_2 > 0$ and $k_0 > 0$, which leads to $\dot{V}(t) < 0$. Thus, virtual dynamics states will converge exponentially to the state of power leader. Using Lemma-1, if auxiliary system in (7) is contracting, then the original

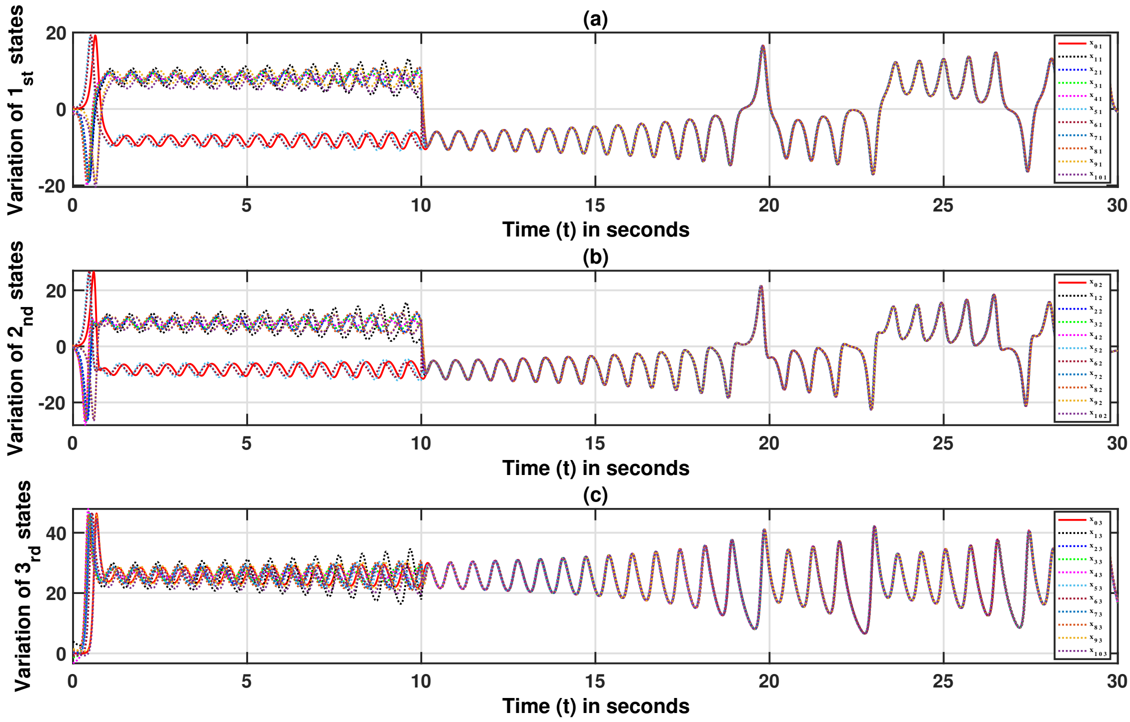

Fig. 2. Synchronization of leader-follower network: (a), (b) and (c) Variation of state trajectories

system trajectories will verify the independent property exponentially. Thus, overall synchronisation of the network is ensured.

IV. Numerical Simulation

To demonstrate the efficacy of the derived analytical results, we investigate the network shown in Fig. 1, which consists of N nodes of nonlinear dynamical systems with an isolated node (power leader). The network consists of two circles, each containing follower nodes, nodes 1 to l_1 represent inner circle, which are directly connected to power leader. Outer circle having nodes from l_1+1 to l_2. Each node is connected bidirectionally to the next node in the circle as well as each node in the outer circle is connected bidirectionally to its corresponding node in the inner circle with delay "d". All the associated coupling gains are considered as symmetrically positive and the bidirectional coupling configuration is assumed between the followers. Time delays are same along the opposite directions i.e. $\tau_{ji} = \tau_{ij}$. The delay between each node on outer circle is represented by "D" symbol. Delays "D" and "d" are not required to be same. Here, $\gamma_i = 1$; for $i = 1, \ldots, l_1$ and 0 otherwise.

Consider a complex dynamical system which consists of 10 similar Lorenz systems with each circle having 5 identical nodes, which are described by

$$\begin{aligned} \dot{\mathbf{x}}_0 &= \mathbf{f}(\mathbf{x}_0, t) \\ \dot{\mathbf{x}}_i &= \mathbf{f}(\mathbf{x}_i, t) + \sum_{j \in \mathscr{N}_i} k_1(\mathbf{x}_j - \mathbf{x}_i) + \gamma_i k_0(\mathbf{x}_0 - \mathbf{x}_i) \\ &\quad + \sum_{j \in \mathscr{H}_i} k_2(\mathbf{x}_j(t-\tau_{ji}) - \mathbf{x}_i) \end{aligned} \tag{11}$$

where $\mathbf{x}_i(t) = (x_{i,1}(t), x_{i,2}(t), x_{i,3}(t))^T$ are state variables, for $i = 1, 2, \ldots, 10$. The dynamics of i_{th} node is given as

$$\begin{aligned} \dot{x}_{i,1} &= -a x_{i,1} + a x_{i,2} + \sum_{j \in \mathscr{H}_i} k_2(\mathbf{x}_{j,1}(t-\tau_{ji}) - \mathbf{x}_{i,1}) \\ &+ \sum_{j \in \mathscr{N}_i} k_1(\mathbf{x}_{j,1} - \mathbf{x}_{i,1}) + k_0(\mathbf{x}_{0,1} - \mathbf{x}_{i,1}) \end{aligned}$$

$$\begin{aligned} \dot{x}_{i,2} &= c x_{i,1} - x_{i,2} - x_{i,1} x_{i,3} + \sum_{j \in \mathscr{H}_i} k_2(\mathbf{x}_{j,2}(t-\tau_{ji}) - \mathbf{x}_{i,2}) \\ &+ \sum_{j \in \mathscr{N}_i} k_1(\mathbf{x}_{j,2} - \mathbf{x}_{i,2}) + k_0(\mathbf{x}_{0,2} - \mathbf{x}_{i,2}) \\ \dot{x}_{i,3} &= x_{i,1} x_{i,2} - b x_{i,3} + \sum_{j \in \mathscr{H}_i} k_2(\mathbf{x}_{j,3}(t-\tau_{ji}) - \mathbf{x}_{i,3}) \\ &+ \sum_{j \in \mathscr{N}_i} k_1(\mathbf{x}_{j,3} - \mathbf{x}_{i,3}) + k_0(\mathbf{x}_{0,3} - \mathbf{x}_{i,3}) \end{aligned} \tag{12}$$

For the specified parameter values of $a = 10$, $b = \frac{8}{3}$, and $c = 28$, respectively, the Lorenz system exhibits chaotic behavior. As we know, for Lorenz system, being chaotic, system states remain bounded. From the chaotic attractor bounds on states can be expressed as $|x_{i,1}| \leq 29$; $|x_{i,2}| \leq 29$; $|x_{i,3}| \leq 57$. The structure of modified Laplacian matrix $\mathscr{L}_{k_1}$ describing the

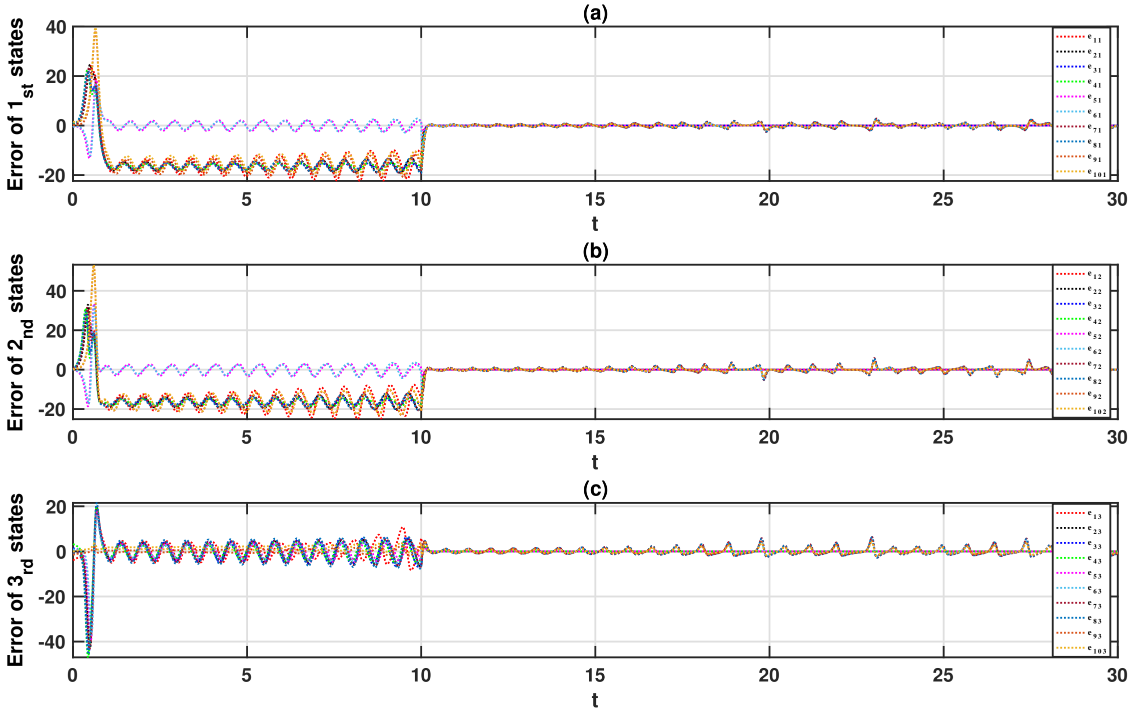

Fig. 3. Synchronization error of leader-follower network: (a), (b) and (c) Variation of error states

interconnections between the inner circle nodes is

$$\mathscr{L}_{k_1} = \begin{bmatrix} \mathbf{2k_1} & -\mathbf{k_1} & \mathbf{0} & \mathbf{0} & -\mathbf{k_1} \\ -\mathbf{k_1} & \mathbf{2k_1} & -\mathbf{k_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{k_1} & \mathbf{2k_1} & -\mathbf{k_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{k_1} & \mathbf{2k_1} & -\mathbf{k_1} \\ -\mathbf{k_1} & \mathbf{0} & -\mathbf{0} & \mathbf{k_1} & \mathbf{2k_1} \end{bmatrix}$$

The expression $\gamma_i k_0 \mathbf{I}_n$ represents a block diagonal matrix with the i_{th} diagonal entry as k_0. Choosing the non-delay and delay coupling coefficients as $k_0 = 10$; $k_1 = 10$ and $k_2 = 10$, respectively, the conditions of Theorem-1 can be easily verified in this case. The communication time delay between inner circle and outer circle nodes is τ_{ji} = 0.02 seconds. The communication time delay between adjacent outer circle nodes is τ_{ji} = 0.001 seconds. The control function is applied after 10 seconds of simulation. The initial conditions for the 10 followers are as follows: $\begin{bmatrix}-0.5 & -0.15 & 4\end{bmatrix}$; $\begin{bmatrix}-0.4 & -0.1 & 0.2\end{bmatrix}$; $\begin{bmatrix}-0.6 & 0.2 & -1\end{bmatrix}$; $\begin{bmatrix}-0.7 & -0.1 & -3\end{bmatrix}$; $\begin{bmatrix}0.7 & -0.4 & 1.2\end{bmatrix}$; $\begin{bmatrix}.71 & -0.6 & -.9\end{bmatrix}$; $\begin{bmatrix}-0.2 & -0.5 & 0.7\end{bmatrix}$; $\begin{bmatrix}-1.5 & 0.6 & 1.02\end{bmatrix}$; $\begin{bmatrix}-0.15 & 0.1 & 1.5\end{bmatrix}$ and $\begin{bmatrix}-1.5 & -0.15 & 0.15\end{bmatrix}$, respectively. The dynamics of the power leader is governed by the Lorenz system with an initial condition of $\begin{bmatrix}0.03 & 0.04 & 0.01\end{bmatrix}$. Using MATLAB, the simulation results of leader follower network are presented in Fig. 2.

A. Synchronisation Error

The synchronisation errors between consecutive systems has also been determined. The error between leader system and follower systems is defined by

$$e_{q,j} = x_{0,j} - x_{q,j}$$

Here, $q = 1, 2, \ldots, 10$; $j = 1, 2, 3$. The control gains are activated at time t = 10 seconds. Fig. 3, shows the variation of synchronisation error between leader and followers for the complex delayed network. The simulation graphs clearly reflect the accomplishment of the objective of synchronisation of the network with a given power leader in presence of delays in node to node interaction.

V. Conclusion

In this paper, control scheme for leader-follower synchronization in a complex network is presented. The network interconnections are assumed to have delays involved in communication links. To derive appropriate coupling gains, contraction theory based methodology is adopted. This research has also focused on the emergence of power leader-follower dynamics, where power leader node can influence the behavior of follower nodes directly or indirectly. The proposed analytical results are validated using a power leader based network with delayed interaction between the adjacent systems. This research topic is an active and evolving field with many potential applications and opportunities for future research. Further, the comparative experiment results will be studied in our future work.

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