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A Reduction based Discrete-Time Model Reference Output Feedback Terminal Sliding Mode Control Approach for SISO Systems

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***Abstract*—This paper proposes a model reference output feedback discrete-time terminal sliding mode (DT-TSMC) approach for SISO systems. The approach is based on a chatter-free equivalent control design that relies on a delay disturbance observer to handle exogenous disturbances. The stability of the proposed is shown via rigorous analysis and it is demonstrated that the term with the fractional power improves the steady state phase of the reference tracking and that an error of $O\left(T^2\right)$ can be achieved. The paper concludes with a simulation example that shows a comparison of the proposed DT-TSMC with a classical discrete-time sliding mode control (DT-SMC) approach. The results show that for a similar transient response, DT-TSMC produces better steady state performance.**

I. Introduction

Sliding mode control (SMC) is is known for its robustness to plant uncertainties and disturbances in continuous time. The rapid development of computer technology and digital controllers resulted in a lot of attention being given to discrete-time sliding mode control (DT-SMC), [1]. To implement such a controller in practice on a digital computer, a discrete time version of the approach is required. Discretisation of a continuous SMC controller is one way to do this. Another approach is to work directly in discrete time, designing the sliding surface and then deriving from it an equivalent control law, to be used as the actual control law. The benefit of the latter approach over the former is the absence of an explicit switching term which causes undesirable chattering, [2]. Chattering is the uninted consequence of the limitation on the switching speed due to the limit on the samling frequency. Several DT-SMC approaches are shown in [2]-[8] that show a tracking error of $O\left(T^2\right)$ is achievable.

In addition to disturbance observers, nonlinear SMC designs have been proposed to further improve the robustness characteristics of DT-SMC, [8]-[11]. One such nonlinear SMC design is the discrete-time terminal sliding mode control (DT-TSMC) which is characterized by the finite-time convergence to the sliding surface and the equilibrium point, which is more advanced than the asymptotic convergence achieved by traditional linear DT-SMC . Subsequently, several DT-TSMC approaces have been proposed such as the DT-TSMC with disturbance observer, [12] and the fast terminal sliding mode (FTSM), [13]. In [14], a terminal smc design that achieves fast chattering-free convergence is presented that combines with receding optimization to deal with constraints so that the sliding mode state follows the reference trajectory predefined by the reaching law.

In this paper a discrete time terminal sliding mode controller is developed utilising only output feedback. The plant dynamics and desired reference model are first expressed in input-output form, and then the error dynamics to be driven to zero in augmented form is obtained. Due to the presence of zeros in the plant dynamics there are many input history terms in the expression. To facilitate the control law design a substitution is used that is inspired by Artsteins model reduction which was originally proposed for control design of continuous time plants with time delay. The substitution yields a dynamics in a new variable that does not contain those input history terms explicitly. The paper is organized as follows: Section 2, gives the problem definition. In Section 3, the main result which includes the DT-TSMC design and stability analysis is presented. In Section 4, a simulation example is presented that shows a comparison of the proposed DT-TSMC approach with classical DT-SMC to demonstrate the effectiveness of the proposed approach. Finally, concluding remarks are given in Section 5.

II. Problem Definition

Consider the continuous-time SISO plant given as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A_{\mathrm{n}}\mathbf{x}(t) + \mathbf{b}_{\mathrm{n}}\left(u + f(t)\right)\\ y(t) &= \mathbf{c}_{\mathrm{n}}\mathbf{x}(t)\end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input and $f(t) \in \mathbb{R}$ is a continuously differentiable disturbance. The matrices $A_{\mathrm{n}} \in \mathbb{R}^{n\times n}$, $\mathbf{b}_{\mathrm{n}} \in \mathbb{R}$ and $\mathbf{c}_{\mathrm{n}} \in \mathbb{R}$ are are the nominal state, input, and output matrices, respectively.

Sampling at uniform time intervals T gives a discrete-time form of the plant (1) described by

$$\begin{aligned}\mathbf{x}_{k+1} &= \Phi\mathbf{x}_k + \boldsymbol{\gamma}u_k + \mathbf{d}_k\\ y_k &= \mathbf{c}_{\mathrm{n}}\mathbf{x}_k\end{aligned} \tag{2}$$

where $\Phi = e^{A_{\mathrm{n}}T}$, $\boldsymbol{\gamma} = \int_0^T \Phi(\tau)\mathbf{b}_{\mathrm{n}}d\tau$ and $\mathbf{d}_k = -\int_0^T \Phi(\tau)\mathbf{b}_{\mathrm{n}}f((k+1)T-\tau)d\tau$. Moreoever, given the fact that $f(t)$ is continuously differentiable, the disturbance $\mathbf{d}_k$ satisfies the following properties:

Property 1: $\mathbf{d}_k = \boldsymbol{\gamma}\nu_k + O\left(T^3\right)$
Property 2: $\|\nu_k - \nu_{k-1}\| \in O(T)$
Property 3: $\|\nu_k - 2\nu_{k-1} + \nu_{k-2}\| \in O\left(T^2\right)$
where $\nu_k = f_k + \frac{1}{2}v_k$, $f_k \triangleq f(kT)$ and $v_k \triangleq v(kT)$ with $v(kT)$ given as $v(kT) = \frac{d}{dt}f(kT)$, respectively.

Assuming the availability of only the output measurement, the plant (2) is written in the input-output form given as

$$\begin{aligned}y_{k+1} =& -\phi_1 y_k - \cdots - \phi_n y_{k-n+1} + \gamma_0\left(u_k + \nu_k\right) + \cdots\\ &+ \gamma_m\left(u_{k-m} + \nu_{k-m}\right) + O\left(T^3\right)\end{aligned} \tag{3}$$

where the parameters $\phi_1, \ldots, \phi_n, \gamma_0, \ldots, \gamma_m$ are obtained from the matrices Φ and $\boldsymbol{\gamma}$ respectively and $n \geq m$. For the plant (3), the following assumptions are made:

Assumption 1: $\phi_n \neq 0$.

Assumption 2: $\phi_1, \ldots, \phi_n \in O(1)$ and $\gamma_0, \ldots, \gamma_m \in O(T)$.

Assumption 3: Open-loop zeros are inside the unit-circle.

Consider, now, the n^{th} order reference model given as

$$y_{\mathrm{m},k+1} = -\phi_{\mathrm{m},1} y_{\mathrm{m},k} - \cdots - \phi_{\mathrm{m},n} y_{\mathrm{m},k-n+1} + \gamma_{\mathrm{m}} r_k \quad (4)$$

where $y_{\mathrm{m},k} \in \mathbb{R}$ is the output of the reference model, $r_k \in \mathbb{R}$ is a reference signal and $\phi_{\mathrm{m},1}, \ldots, \phi_{\mathrm{m},n}$ are selected such that (4) is Schur stable. The control objective is to find a bounded input u_k which will drive the plant output y_k to track the output of a reference model output $y_{\mathrm{m},k}$ asymptotically, i.e. $\lim_{k\to\infty} |y_k - y_{\mathrm{m},k}| = O\left(T^2\right)$ in the presence of the disturbance ν_k.

III. Main Result

In this section, the design of the delay disturbance observer and the control law is presented which is then concluded by a rigorous stability analysis of the closed-loop system.

A. Disturbance Observer

Consider the plant (3), a form of the delay disturbance observer proposed in [2] is obtained by rewriting (3) as

$$\begin{aligned}\nu_{k-1} + &\frac{\gamma_1}{\gamma_0}\nu_{k-2} + \cdots + \frac{\gamma_m}{\gamma_0}\nu_{k-m-1} \\ =&\frac{1}{\gamma_0}\left(y_k + \phi_1 y_{k-1} + \cdots + \phi_n y_{k-n}\right) - u_{k-1} - \cdots \\ &- \frac{\gamma_m}{\gamma_0} u_{k-m-1} + O\left(T^3\right) \end{aligned} \quad (5)$$

and lumping all the terms on the left-hand-side such that

$$\begin{aligned}\rho_{k-1} =&\frac{1}{\gamma_0}\left(y_k + \phi_1 y_{k-1} + \cdots + \phi_n y_{k-n}\right) - u_{k-1} - \cdots \\ &- \frac{\gamma_m}{\gamma_0} u_{k-m-1} + O\left(T^3\right) \end{aligned} \quad (6)$$

Substituting the actual disturbance ρ_k such that $\rho_{k-1} = \hat{\rho}_k$, where $\hat{\rho}_k$ is the estimate of ρ_k and ignoring the $O\left(T^3\right)$ term, it is obtained that

$$\begin{aligned}\hat{\rho}_k =&\frac{1}{\gamma_0}\left(y_k + \phi_1 y_{k-1} + \cdots + \phi_n y_{k-n}\right) - u_{k-1} - \cdots \\ &- \frac{\gamma_m}{\gamma_0} u_{k-m-1} \end{aligned} \quad (7)$$

which is the delay disturbance observer for the disturbance ρ_k as the estimate for ρ_k is the delayed value ρ_{k-1}. This is evident when comparing (7) with the dynamics (6).

Remark 1: To achieve an $O\left(T^3\right)$ disturbance compensation, the term $-2\hat{\rho}_k + \hat{\rho}_{k-1}$ will be added to the control input $u(k)$ such that the disturbance term in the closed-loop system will be reduced to the form $\rho_k - 2\hat{\rho}_k + \hat{\rho}_{k-1}$ which is nothing but the second-order backward difference and, according to *Property 4*, is given as $\rho_k - 2\rho_{k-1} + \rho_{k-2} \in O\left(T^2\right)$.

B. Discrete-Time Terminal Sliding Mode Controller Design

To proceed with the DT-TSMC controller design, define the tracking error as $e_k = y_k - y_{\mathrm{m},k}$ such that substituting (3) and (4) gives the tracking error dynamics

$$\begin{aligned} e_{k+1} =& -\phi_1 e_k - \cdots - \phi_n e_{k-n+1} + \gamma_0 u_k + \cdots + \gamma_m u_{k-m} \\ &+ \tilde{\phi}_1 y_{\mathrm{m},k} + \cdots + \tilde{\phi}_n y_{\mathrm{m},k-n+1} - \gamma_{\mathrm{m}} r_k + \gamma_0 \rho_k \\ &+ O\left(T^3\right) \end{aligned} \quad (8)$$

where $\tilde{\phi}_i \triangleq \phi_{\mathrm{m},i} - \phi_i \; \forall i = [1, n]$, respectively. Let $\varrho_k = \rho_k + \frac{1}{\gamma_0}\left(\tilde{\phi}_1 y_{\mathrm{m},k} + \cdots + \tilde{\phi}_n y_{\mathrm{m},k-n+1} - \gamma_{\mathrm{m}} r_k\right)$ and writing the error dynamics (8) in augmented form, it is obtained that

$$\bar{\mathbf{e}}_{k+1} = \Theta \bar{\mathbf{e}}_k + \bar{\boldsymbol{\gamma}}_0 u_k + \cdots + \bar{\boldsymbol{\gamma}}_m u_{k-m} + \bar{\boldsymbol{\gamma}}_0 \varrho_k + \boldsymbol{\delta}_k \quad (9)$$

where $\bar{\mathbf{e}}_k^\top \triangleq \begin{bmatrix} e_{k-n+1} & \cdots & e_k \end{bmatrix} \in \mathbb{R}^n$ is the augmented output vector, $\bar{\boldsymbol{\gamma}}_i^\top \triangleq \begin{bmatrix} 0 & \cdots & 0 & \gamma_i \end{bmatrix} \in \mathbb{R}^n \; \forall i = [0, m]$ and $\|\boldsymbol{\delta}_k\| \in O\left(T^3\right)$ is the approximation error in *Property 1*. The state matrix $\Theta \in \mathbb{R}^{n\times n}$ is defined as

$$\Theta \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -\phi_n & \cdots & \cdots & -\phi_1 \end{bmatrix}$$

Remark 2: The pairs $(\Theta, \bar{\boldsymbol{\gamma}}_0), \ldots, (\Theta, \bar{\boldsymbol{\gamma}}_m)$ are in controllable form and are, therefore, controllable.

Consider, now, the new state vector given as

$$\mathbf{z}_{k+1} = \bar{\mathbf{e}}_{k+1} + \sum_{j=0}^{m-1} \boldsymbol{\vartheta}_j \left(u_{k-j} + \chi_{k-j}\right) \quad (10)$$

where $\mathbf{z}_k \in \mathbb{R}^n$ and $\boldsymbol{\vartheta}_j \in \mathbb{R}^n \; \forall j = [0, m-1]$ are parameter vectors that will be computed interms of Θ and $\bar{\boldsymbol{\gamma}}_i \; \forall i = [1, m]$. The term χ_k will be derived such that the tracking error $\lim_{k\to\infty} e_k = O\left(T^2\right)$. Substitution of (9) in (10) and setting $\boldsymbol{\vartheta}_j = \sum_{i=j}^{m-1} \Theta^{j-i-1} \bar{\boldsymbol{\gamma}}_{i+1}$, it is obtained that

$$\begin{aligned}\mathbf{z}_{k+1} =& \Theta \mathbf{z}_k + \boldsymbol{\beta} u_k + \bar{\boldsymbol{\gamma}}_0 \varrho_k + \boldsymbol{\delta}_k + \boldsymbol{\vartheta}_0 \chi_k + \Big(\boldsymbol{\vartheta}_1 - \Theta \boldsymbol{\vartheta}_0\Big) \\ &\times \chi_{k-1} + \cdots + \Big(\boldsymbol{\vartheta}_{m-1} - \Theta \boldsymbol{\vartheta}_{m-2}\Big)\chi_{k-m+1} \\ &- \Theta \boldsymbol{\vartheta}_{m-1} \chi_{k-m} \end{aligned} \quad (11)$$

where $\boldsymbol{\beta} = \sum_{i=0}^{m} \Theta^{-i} \bar{\boldsymbol{\gamma}}_i$. Adding and subtracting the term $\bar{\boldsymbol{\gamma}}_0 \chi_k$ on the right-hand-side of (11), it is obtained that

$$\begin{aligned}\mathbf{z}_{k+1} =& \Theta \mathbf{z}_k + \boldsymbol{\beta} u_k + \bar{\boldsymbol{\gamma}}_0 \varrho_k + \boldsymbol{\delta}_k + \left(\bar{\boldsymbol{\gamma}}_0 + \boldsymbol{\vartheta}_0\right)\chi_k - \bar{\boldsymbol{\gamma}}_0 \chi_k \\ &+ \Big(\boldsymbol{\vartheta}_1 - \Theta \boldsymbol{\vartheta}_0\Big)\chi_{k-1} + \cdots + \Big(\boldsymbol{\vartheta}_{m-1} - \Theta \boldsymbol{\vartheta}_{m-2}\Big) \\ &\times \chi_{k-m+1} - \Theta \boldsymbol{\vartheta}_{m-1} \chi_{k-m} \end{aligned} \quad (12)$$

Using the definitions of $\boldsymbol{\beta}$ and $\boldsymbol{\vartheta}_j$, (23) is simplified as

$$\begin{aligned}\mathbf{z}_{k+1} =& \Theta \mathbf{z}_k + \boldsymbol{\beta} u_k + \bar{\boldsymbol{\gamma}}_0 \varrho_k + \boldsymbol{\delta}_k + \boldsymbol{\beta} u_k \chi_k - \bar{\boldsymbol{\gamma}}_0 \chi_k \\ &- \bar{\boldsymbol{\gamma}}_1 \chi_{k-1} - \cdots - \bar{\boldsymbol{\gamma}}_m \chi_{k-m} \\ =& \Theta \mathbf{z}_k + \boldsymbol{\beta}\left(u_k + \chi_k\right) + \boldsymbol{\delta}_k + \boldsymbol{\epsilon}_k \end{aligned} \quad (13)$$

where $\boldsymbol{\epsilon}_k = \bar{\boldsymbol{\gamma}}_0 \varrho_k - \bar{\boldsymbol{\gamma}}_0 \chi_k - \cdots - \bar{\boldsymbol{\gamma}}_m \chi_{k-m}$.

Remark 3: According to *Assumption 1*, $\phi_n \neq 0$ and, therefore, Θ is non-singular.

Lemma 1: The dynamics (11) is controllable, i.e., the pair $\Theta, \boldsymbol{\beta}$ is controllable.

Proof: Consider the controllability matrix, for the pair $\Theta, \boldsymbol{\beta}$, given as

$$\begin{aligned} W_{\rm c} &= \left[\boldsymbol{\beta} \mid \cdots \mid \Theta^{n-1}\boldsymbol{\beta}\right] = \left[\sum_{i=0}^{m}\Theta^{-i}\bar{\boldsymbol{\gamma}}_i \mid \cdots \right.\\ &\quad \left. \mid \Theta^{n-1}\sum_{i=0}^{m}\Theta^{-i}\bar{\boldsymbol{\gamma}}_i\right] \\ &= \left[\bar{\boldsymbol{\gamma}}_0 \mid \cdots \mid \Theta^{n-1}\bar{\boldsymbol{\gamma}}_0\right] + \Theta^{-1}\left[\bar{\boldsymbol{\gamma}}_1 \mid \cdots \mid \Theta^{n-1}\bar{\boldsymbol{\gamma}}_1\right] \\ &\quad + \cdots + \Theta^{-m}\left[\bar{\boldsymbol{\gamma}}_m \mid \cdots \mid \Theta^{n-1}\bar{\boldsymbol{\gamma}}_m\right] \\ &= \left[I + \tfrac{\gamma_1}{\gamma_0}\Theta^{-1} + \cdots + \tfrac{\gamma_1}{\gamma_0}\Theta^{-m}\right]\left[\bar{\boldsymbol{\gamma}}_0 \mid \cdots \mid \Theta^{n-1}\bar{\boldsymbol{\gamma}}_0\right] \\ &= \Theta^{-m}\left[\Theta^m + \tfrac{\gamma_1}{\gamma_0}\Theta^{m-1} + \cdots + \tfrac{\gamma_1}{\gamma_0}I\right]W_{\rm c,0} \end{aligned} \tag{14}$$

where $W_{\rm c,0} = \left[\bar{\boldsymbol{\gamma}}_0 \mid \cdots \mid \Theta^{n-1}\bar{\boldsymbol{\gamma}}_0\right] \in \mathbb{R}^{n\times n}$ is non-singular since $\Theta, \bar{\boldsymbol{\gamma}}_0$ is a controllable pair. Let S be a non-singular transformation matrix such that $\Theta_{\rm d} = S^{-1}\Theta S$ where $\Theta_{\rm d}$ is the diagonal (or Jordan) matrix of the eigenvalues of Θ. Substituting in (14), it is obtained that

$$\begin{aligned} \Theta^m W_{\rm c} &= \left[S\Theta_{\rm d}^m S^{-1} + \tfrac{\gamma_1}{\gamma_0}S\Theta_{\rm d}^{m-1}S^{-1} + \cdots + \tfrac{\gamma_1}{\gamma_0}SS^{-1}\right]W_{\rm c,0} \\ &= S\left[\Theta_{\rm d}^m + \tfrac{\gamma_1}{\gamma_0}\Theta_{\rm d}^{m-1} + \cdots + \tfrac{\gamma_1}{\gamma_0}I\right]S^{-1}W_{\rm c,0} \end{aligned} \tag{15}$$

Recall that the eigenvalues of Θ and the roots of the polynomial $\left(\gamma_0 q^n + \gamma_1 q^{m-1} + \cdots + \gamma_1\right)$ are the open-loop poles and open-loop zeros of the plant (3), respectively. If the plant (3) is in the proper form then there are no pole-zero cancellations in the open-loop. This implies that $eig(\Theta) \neq roots\left(\gamma_0 q^n + \gamma_1 q^{m-1} + \cdots + \gamma_1\right)$ and, as a result, $\Theta_{\rm d}^m + \frac{\gamma_1}{\gamma_0}\Theta_{\rm d}^{m-1} + \cdots + \frac{\gamma_1}{\gamma_0}I$ is a diagonal (or upper triangular) matrix of non-zero diagonal elements. Therefore, $\Theta_{\rm d}^m + \frac{\gamma_1}{\gamma_0}\Theta_{\rm d}^{m-1} + \cdots + \frac{\gamma_1}{\gamma_0}I$ is non-singular and, thus, $\Theta^m + \frac{\gamma_1}{\gamma_0}\Theta^{m-1} + \cdots + \frac{\gamma_1}{\gamma_0}I$ is also non-singular. Since, all the matrices on the right-hand-side of (15) are non-singular and Θ is non-singular then $W_{\rm c}$ is non-singular and the pair $\Theta, \boldsymbol{\beta}$ is controllable. ■

With the controllability of the system (11) established in (Lemma 1), it is now possible to derive the DT-TSMC control law. Consider the sliding surface, $\sigma_k \in \mathbb{R}$, given as

$$\sigma_k = \bar{d}\mathbf{z}_k + \bar{\alpha}\|\mathbf{z}_{k-1}\|^{p-1}\mathbf{z}_{k-1} \tag{16}$$

where $\bar{d} \in \mathbb{R}^n$, $\bar{\alpha} \in \mathbb{R}^n$ and $p \in \mathbb{R}$ are the controller parameters. The exponent $p < 1$ is a fraction. Setting $\sigma_{k+1} = 0$ and substituting (11), it is obtained that

$$\begin{aligned} \sigma_{k+1} &= \bar{d}\mathbf{z}_{k+1} + \bar{\alpha}\|\mathbf{z}_k\|^{p-1}\mathbf{z}_k \\ &= \bar{d}\left(\Theta\mathbf{z}_k + \boldsymbol{\beta}u_k + \boldsymbol{\beta}\chi_k + \boldsymbol{\delta}_k + \boldsymbol{\epsilon}_k\right) + \bar{\alpha}\|\mathbf{z}_k\|^{p-1}\mathbf{z}_k \\ &= 0 \end{aligned} \tag{17}$$

Ignoring the term $\boldsymbol{\delta}_k + \boldsymbol{\epsilon}_k$ and deriving an expression for the control law, it is obtained that

$$u_k = -\left(\bar{d}\boldsymbol{\beta}\right)^{-1}\left[\bar{d}\Theta\mathbf{z}_k + \bar{\alpha}\|\mathbf{z}_k\|^{p-1}\mathbf{z}_k\right] - \chi_k \tag{18}$$

Substitution of the control law (18) in (17) results in a bound on σ_k given as

$$\sigma_{k+1} = \bar{d}\left(\boldsymbol{\delta}_k + \boldsymbol{\epsilon}_k\right) \tag{19}$$

Consider now the definition of $\boldsymbol{\epsilon}_k$, since, the disturbance measurement is not available it is obtained that

$$\begin{aligned} \boldsymbol{\epsilon}_k &= \bar{\boldsymbol{\gamma}}_0\varrho_k - \bar{\boldsymbol{\gamma}}_0\chi_k - \cdots - \bar{\boldsymbol{\gamma}}_m\chi_{k-m} \\ &= \bar{\boldsymbol{\gamma}}_0\varrho_k - \bar{\boldsymbol{\gamma}}_0\hat{\varrho}_k + \bar{\boldsymbol{\gamma}}_0\hat{\varrho}_k - \bar{\boldsymbol{\gamma}}_0\chi_k - \cdots - \bar{\boldsymbol{\gamma}}_m\chi_{k-m} \end{aligned} \tag{20}$$

where $\hat{\varrho}_k = 2\hat{\rho}_k - \hat{\rho}_{k-1} + \frac{1}{\gamma_0}\left(\tilde{\phi}_1 y_{{\rm m},k} + \cdots + \tilde{\phi}_n y_{{\rm m},k-n+1}\right.$ $\left.+\gamma_{\rm m}r_k\right)$. Note that $\hat{\varrho}_k$ is defined such that $\varrho_k - \hat{\varrho}_k = \rho_k - 2\hat{\rho}_k - \hat{\rho}_{k-1} = \rho_k - 2\rho_{k-1} - \rho_{k-2} \in O\left(T^2\right)$. Futhermore, if χ_k is selected such that dynamics $\chi_k + \frac{\gamma_1}{\gamma_0}\chi_{k-1} + \cdots + \frac{\gamma_m}{\gamma_0}\bar{\boldsymbol{\gamma}}_m\chi_{k-m} = \hat{\varrho}_k$ is satisified, it is obtained that

$$\begin{aligned} \sigma_{k+1} &= \bar{d}\left(\boldsymbol{\delta}_k + \boldsymbol{\epsilon}_k\right) = \bar{d}\left(\boldsymbol{\delta}_k + \boldsymbol{\epsilon}_k\right) \\ &= \bar{d}\left(\boldsymbol{\delta}_k + \bar{\boldsymbol{\gamma}}_0\left(\rho_k - 2\rho_{k-1} - \rho_{k-2}\right)\right) \in O\left(T^3\right) \end{aligned} \tag{21}$$

where $\bar{d}$ is selected such that $\left\|\bar{d}\right\| \in O(1)$. This bound is similar to that shown in [2].

Remark 4: The selection of the controller parameter $\bar{d}$ follows the usual approaches such as that presented in [15] while ensuring that $\bar{d}\boldsymbol{\beta} > 0$. The selection process for the parameter $\bar{\alpha}$ and p will be presented in the next subsection.

C. Stability Analysis

In this subsection, it is shown that the state $\mathbf{z}_k$ converges asymptotically (*Lemma 2*) and that the proposed DT-TSMC control law drives the plant output y_k to a bound of $O\left(T^2\right)$ asymptotically (*Theorem 1*).

Lemma 2: With the proper selection of the controller parameters $\bar{d}$, $\bar{\alpha}$ and p, the following is true:

$$\lim_{k\to\infty}\|\mathbf{z}_k\| \in O\left(T^2\right) \tag{22}$$

Proof: Consider the system (11), with the substitution of the control law (18) it is obtained that

$$\begin{aligned} \mathbf{z}_{k+1} &= \Theta\mathbf{z}_k + \boldsymbol{\beta}\left(u_k + \chi_k\right) + \boldsymbol{\varepsilon}_k \\ &= \left(\Theta - \boldsymbol{\beta}\left(\bar{d}\boldsymbol{\beta}\right)^{-1}\bar{d}\Theta\right)\mathbf{z}_k - \boldsymbol{\beta}\left(\bar{d}\boldsymbol{\beta}\right)^{-1}\bar{\alpha}\|\mathbf{z}_k\|^{p-1}\mathbf{z}_k \\ &\quad + \boldsymbol{\varepsilon}_k \\ &= \Theta_{\rm m}\mathbf{z}_k - \boldsymbol{\beta}\left(\bar{d}\boldsymbol{\beta}\right)^{-1}\bar{\alpha}\|\mathbf{z}_k\|^{p-1}\mathbf{z}_k + \boldsymbol{\varepsilon}_k \\ &= \Theta_{\rm m}\mathbf{z}_k - \lambda\boldsymbol{\beta}\bar{\alpha}\|\mathbf{z}_k\|^{p-1}\mathbf{z}_k + \boldsymbol{\varepsilon}_k \end{aligned} \tag{23}$$

where $\boldsymbol{\varepsilon}_k = \boldsymbol{\delta}_k + \boldsymbol{\epsilon}_k$, $\Theta_{\rm m} \triangleq \left(\Theta - \lambda\boldsymbol{\beta}\bar{d}\Theta\right)$ and $\lambda \triangleq \left(\bar{d}\boldsymbol{\beta}\right)^{-1}$ with $\bar{d}$ selected such that $\Theta_{\rm m}$ is Schur stable. Now, consider the positive function

$$V_k = \mathbf{z}_k^\top P\mathbf{z}_k \tag{24}$$

where $P \in \mathbb{R}^n$ is some symmetric positive-definite matrix. The forward difference ΔV_k is

$$\begin{aligned} \Delta V_k &= V_{k+1} - V_k = \mathbf{z}_{k+1}^\top P\mathbf{z}_{k+1} - \mathbf{z}_k^\top P\mathbf{z}_k \\ &= \left(\Theta_{\rm m}\mathbf{z}_k - \lambda\boldsymbol{\beta}\bar{\alpha}\|\mathbf{z}_k\|^{p-1}\mathbf{z}_k + \boldsymbol{\varepsilon}_k\right)^\top P\left(\Theta_{\rm m}\mathbf{z}_k\right. \\ &\quad \left.-\lambda\boldsymbol{\beta}\bar{\alpha}\|\mathbf{z}_k\|^{p-1}\mathbf{z}_k + \boldsymbol{\varepsilon}_k\right) - \mathbf{z}_k^\top P\mathbf{z}_k \\ &= \mathbf{z}_k^\top\left(\Theta_{\rm m}^\top P\Theta_{\rm m} - P\right)\mathbf{z}_k - 2\lambda\|\mathbf{z}_k\|^{p-1}\mathbf{z}_k^\top\bar{\alpha}^\top\boldsymbol{\beta}^\top P \\ &\quad \times\left(\Theta_{\rm m}\mathbf{z}_k + \boldsymbol{\varepsilon}_k\right) + 2\mathbf{z}_k^\top\Theta_{\rm m}P\boldsymbol{\varepsilon}_k + \boldsymbol{\varepsilon}_k^\top P\boldsymbol{\varepsilon}_k \\ &\quad + \lambda^2\|\mathbf{z}_k\|^{2p-2}\mathbf{z}_k^\top\bar{\alpha}^\top\boldsymbol{\beta}^\top\boldsymbol{\beta}\bar{\alpha}\mathbf{z}_k \end{aligned} \tag{25}$$

Using $Q = -\left(\Theta_{\mathrm{m}}^\top P \Theta_{\mathrm{m}} - P\right) \in \mathbb{R}^n$ which is positive-definite since Θ_{m} is Schur stable and setting $\bar{\alpha} = \frac{1}{\rho}\boldsymbol{\beta}^\top P\Theta_{\mathrm{m}}$ for some $\rho > 0 \in \mathbb{R}$, (25) is simplified as

$$\begin{aligned}\Delta V_k = &- \mathbf{z}_k^\top \Bigg(Q + 2\frac{\lambda\rho}{\|\mathbf{z}_k\|^{1-p}}\bar{\alpha}^\top\bar{\alpha} + 2\frac{\lambda}{\|\mathbf{z}_k\|^{3-p}}\bar{\alpha}^\top\boldsymbol{\beta}^\top \\ &\times P\boldsymbol{\varepsilon}_k\mathbf{z}_k^\top - \frac{\lambda^2}{\|\mathbf{z}_k\|^{2-2p}}\bar{\alpha}^\top\boldsymbol{\beta}^\top\boldsymbol{\beta}\bar{\alpha} - \frac{2}{\|\mathbf{z}_k\|^2}\Theta_{\mathrm{m}}P \\ &\times \boldsymbol{\varepsilon}_k\mathbf{z}_k^\top - \frac{\boldsymbol{\varepsilon}_k^\top P\boldsymbol{\varepsilon}_k}{\|\mathbf{z}_k\|^4}\mathbf{z}_k\mathbf{z}_k^\top\Bigg)\mathbf{z}_k \end{aligned} \tag{26}$$

Note that in (26), if $\|\mathbf{z}_k\| \geq \frac{1}{\mu}\|\boldsymbol{\varepsilon}_k\| \in O\left(T^2\right)$ where μ is selected such that $\mu > 0 \in O(T)$, then from the terms that include $\boldsymbol{\varepsilon}_k$ in (26), an inequality is obtained as

$$\begin{aligned}&\mathbf{z}_k^\top\Bigg(\frac{2}{\|\mathbf{z}_k\|^2}\Theta_{\mathrm{m}}P\boldsymbol{\varepsilon}_k\mathbf{z}_k^\top + \frac{\boldsymbol{\varepsilon}_k^\top P\boldsymbol{\varepsilon}_k}{\|\mathbf{z}_k\|^4}\mathbf{z}_k\mathbf{z}_k^\top - \frac{2\lambda}{\|\mathbf{z}_k\|^{3-p}}\bar{\alpha}^\top\boldsymbol{\beta}^\top \\ &\quad\times P\boldsymbol{\varepsilon}_k\mathbf{z}_k^\top\Bigg)\mathbf{z}_k \leq \Bigg(\frac{2\left\|\Theta_{\mathrm{m}}P\right\|\left\|\boldsymbol{\varepsilon}_k\right\|\left\|\mathbf{z}_k\right\|}{\|\mathbf{z}_k\|^2} + \frac{\|P\|}{\|\mathbf{z}_k\|^4} \\ &\quad\times\|\boldsymbol{\varepsilon}_k\|^2\|\mathbf{z}_k\|^2 + 2\frac{\lambda\left\|\bar{\alpha}^\top\boldsymbol{\beta}^\top P\right\|\left\|\boldsymbol{\varepsilon}_k\right\|\left\|\mathbf{z}_k\right\|}{\|\mathbf{z}_k\|^{3-p}}\Bigg)\mathbf{z}_k^\top\mathbf{z}_k \\ &\quad\leq \mu\left(\mu + 2\left(\|\Theta_{\mathrm{m}}\| + \frac{\lambda\left\|\bar{\alpha}\boldsymbol{\beta}\right\|}{\|\mathbf{z}_k\|^{1-p}}\right)\right)\|P\|\,\mathbf{z}_k^\top\mathbf{z}_k \end{aligned} \tag{27}$$

Substitution of (27) in (26), it is obtained that

$$\begin{aligned}\Delta V_k \leq &- \mathbf{z}_k^\top\Bigg(Q + \frac{2\lambda}{\|\mathbf{z}_k\|^{1-p}}\left(\rho\bar{\alpha}^\top\bar{\alpha} - \mu\left\|\bar{\alpha}\boldsymbol{\beta}\right\|\|P\|\,I\right) \\ &- \frac{\lambda^2\boldsymbol{\beta}^\top\boldsymbol{\beta}}{\|\mathbf{z}_k\|^{2-2p}}\bar{\alpha}^\top\bar{\alpha} - \mu\big(2\left\|\Theta_{\mathrm{m}}\right\| + \mu\big)\|P\|I\Bigg)\mathbf{z}_k \end{aligned} \tag{28}$$

Let $Q = Q_1 + Q_2$ and $\rho = \rho_1 + \rho_2$ for some positive-definite Q_1,Q_2 and $\rho_1, \rho_2 > 0$ such that

$$Q_2 = \mu\big(2\left\|\Theta_{\mathrm{m}}\right\| + \mu\big)\|P\|\,I \tag{29}$$

and

$$\rho_2\bar{\alpha}^\top\bar{\alpha} - \mu\left\|\bar{\alpha}\boldsymbol{\beta}\right\|\|P\|\,I \geq 0 \tag{30}$$

Substituting in (28), it is obtained that

$$\Delta V_k \leq -\mathbf{z}_k^\top\left(Q + \lambda\varphi\left(\|\mathbf{z}_k\|\right)\bar{\alpha}^\top\bar{\alpha}\right)\mathbf{z}_k \tag{31}$$

where $\varphi\left(\|\mathbf{z}_k\|\right)$ is defined as

$$\varphi\left(\|\mathbf{z}_k\|\right) \triangleq \left(\frac{2\rho_1}{\|\mathbf{z}_k\|^{1-p}} - \frac{\lambda\boldsymbol{\beta}^\top\boldsymbol{\beta}}{\|\mathbf{z}_k\|^{2-2p}}\right) \tag{32}$$

Note that the maximum positive value of the function $\varphi\left(\|\mathbf{z}_k\|\right)$ is computed by solving

$$\frac{\partial}{\partial\|\mathbf{z}_k\|}\varphi\left(\|\mathbf{z}_k\|\right) = \frac{1}{\|\mathbf{z}_k\|^{2-p}}\left(\rho_1 - \frac{\lambda\boldsymbol{\beta}^\top\boldsymbol{\beta}}{\|\mathbf{z}_k\|^{1-p}}\right) = 0 \tag{33}$$

which gives a non-trivial solution as

$$\|\mathbf{z}_k\| = \left(\frac{\lambda\boldsymbol{\beta}^\top\boldsymbol{\beta}}{\rho_1}\right)^{\frac{1}{1-p}} \tag{34}$$

Futhermore, the function $\varphi\left(\|\mathbf{z}_k\|\right)$ changes sign if

$$\|\mathbf{z}_k\| \leq \left(\frac{\lambda\boldsymbol{\beta}^\top\boldsymbol{\beta}}{2\rho_1}\right)^{\frac{1}{1-p}} \tag{35}$$

Therefore, ρ_1 and p can be selected such that the maximum gain occurs as $\|\mathbf{z}_k\|$ approaches the $O\left(T^2\right)$ bound while the sign change of $\varphi\left(\|\mathbf{z}_k\|\right)$ occurs below the $O\left(T^2\right)$. ■

Theorem 1: The output tracking error of the closed-loop system approaches a bound of $O\left(T^2\right)$ asymptotically, i.e. $\lim_{k\to\infty}|e_k| \leq O\left(T^2\right)$.

Proof: Consider the expression (10), with the substitution of the control law (18) it is obtained that

$$\begin{aligned}\mathbf{z}_{k+1} &= \bar{\mathbf{e}}_{k+1} + \sum_{j=0}^{m-1}\boldsymbol{\vartheta}_j\left(u_{k-j} + \chi_{k-j}\right) \\ &= \bar{\mathbf{e}}_{k+1} - \lambda\sum_{j=0}^{m-1}\boldsymbol{\vartheta}_j\left(\bar{d}\Theta\mathbf{z}_{k-j} + \bar{\alpha}\|\mathbf{z}_{k-j}\|^{p-1}\mathbf{z}_{k-j}\right)\end{aligned} \tag{36}$$

From *Lemma* 2, the parameters ρ_1 and p are selected such that (34) is true within the vicinity of $O\left(T^2\right)$. Therefore, substitution of (34) in (36), it is obtained that

$$\begin{aligned}\bar{\mathbf{e}}_{k+1} &= \mathbf{z}_{k+1} + \lambda\sum_{j=0}^{m-1}\boldsymbol{\vartheta}_j\left(\bar{d}\Theta + \frac{\rho_1}{\boldsymbol{\beta}^\top\boldsymbol{\beta}}\bar{\alpha}\right)\mathbf{z}_{k-j} \\ &= \mathbf{z}_{k+1} + \lambda\sum_{j=0}^{m-1}\boldsymbol{\vartheta}_j\left(\bar{d}\Theta + \frac{\rho_1}{\rho\boldsymbol{\beta}^\top\boldsymbol{\beta}}\boldsymbol{\beta}^\top P\Theta_{\mathrm{m}}\right)\mathbf{z}_{k-j} \\ &= \mathbf{z}_{k+1} + \lambda\sum_{j=0}^{m-1}\boldsymbol{\vartheta}_j\left(\bar{d}\Theta + \frac{\rho_1}{\rho\|\boldsymbol{\beta}\|^2}\boldsymbol{\beta}^\top P\Theta_{\mathrm{m}}\right)\mathbf{z}_{k-j}\end{aligned} \tag{37}$$

and e_{k+1} is obtained as

$$\begin{aligned}e_{k+1} = &\,\mathbf{c}\mathbf{z}_{k+1} + \lambda\mathbf{c}\sum_{j=0}^{m-1}\boldsymbol{\vartheta}_j\bar{d}\Theta\mathbf{z}_{k-j} + \frac{\rho_1}{\rho\|\boldsymbol{\beta}\|^2}\lambda\mathbf{c}\sum_{j=0}^{m-1}\boldsymbol{\vartheta}_j \\ &\times\boldsymbol{\beta}^\top P\Theta_{\mathrm{m}}\mathbf{z}_{k-j}\end{aligned} \tag{38}$$

where $\mathbf{c} = \begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}$. The bound on y_k is obtained as

$$\begin{aligned}|e_{k+1}| \leq &\,\|\mathbf{z}_{k+1}\| + \frac{\rho_1\|P\|\left\|\Theta_{\mathrm{m}}\right\|}{\rho\|\boldsymbol{\beta}\|}\sum_{j=0}^{m-1}\|\boldsymbol{\vartheta}_j\|\left\|\mathbf{z}_{k-j}\right\| \\ &+ \lambda\|\Theta\|\sum_{j=0}^{m-1}\left\|\boldsymbol{\vartheta}_j\bar{d}\right\|\left\|\mathbf{z}_{k-j}\right\|\end{aligned} \tag{39}$$

From *Assumption* 2, $\|\Theta\| \in O(1)$ and $\left\|\Theta^{-1}\right\| \in O(1)$. Furthermore, since $\boldsymbol{\beta} = \sum_{i=0}^{m}\Theta^{-i}\bar{\gamma}_i$ and $\boldsymbol{\vartheta}_j = \sum_{i=j}^{m-1}\Theta^{j-i-1}\bar{\gamma}_{i+1}$ then $\|\boldsymbol{\beta}\|, \|\boldsymbol{\vartheta}_j\| \in O(T)$. Finally, if ρ,

ρ_1, P and Θ_{m} are selected such that $O(\rho) = O(\rho_1)$ and $\|P\|, \|\Theta_{\mathrm{m}}\| \in O(1)$ then it is obtained that

$$|e_{k+1}| \leq \|\mathbf{z}_{k+1}\| + \frac{O(1)}{O(T)} \cdot O(T) \cdot O(1) \cdot O(1) \|\mathbf{z}_k\| + O\left(\frac{1}{T}\right) \cdot O(T) \cdot O(1) \|\mathbf{z}_k\| \tag{40}$$

and, since, $\lim_{k\to\infty} \|\mathbf{z}_k\| \in O\left(T^2\right)$ then

$$\lim_{k\to\infty} |e_k| \leq O\left(T^2\right) \tag{41}$$

■

IV. Simulation Example

In this section, a simulation example is presented that shows a comparison between the proposed DT-TSMC approach and classical non-chattering DT-SMC. The disturbance observer (7) will be utilized in both approaches so that the comparison will highlight the advantages of the term with the fractional power.

Consider the system represented as follows

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 8.6 \times 10^{-3} \\ 2.8 \\ 20 \end{bmatrix} (u(t) + f(t))$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) \tag{42}$$

where the disturbance signal is given as $f(t) = 10\sin(2\pi t)$. Sampling the system at an uniform sampling interval of $T = 0.01$s gives a sampled-data input-output model given as

$$y_{k+1} = 3y_k - 3y_{k-1} + y_{k-2} + 0.001(u_k + \nu_k) - 0.0017 \times (u_{k-1} + \nu_{k-1}) + 0.0007(u_{k-2} + \nu_{k-2}) \tag{43}$$

The reference model is selected as

$$y_{\mathrm{m},k+1} = 1.1y_{\mathrm{m},k} - 0.19y_{\mathrm{m},k-1} + 0.009y_{\mathrm{m},k-2} + 0.081r_k \tag{44}$$

where r_k profile can be seen in Fig.1. The DT-TSMC parameters are selected as

$$\bar{d}^{\top} = \begin{bmatrix} -1 \\ -1 \\ 2.1 \end{bmatrix}, \; \alpha^{\top} = \begin{bmatrix} -1.4688 \\ 3.0121 \\ -1.4756 \end{bmatrix}, \; p = \frac{4.3}{5}, \; \rho_1 = \rho_2 = 0.1$$

For the conventional DT-SMC, the controller parameter is selected as $\bar{d} = [-1 \;\; -1 \;\; 2.2]$ with the control law given as

$$u_k = -\lambda \bar{d} \Theta \mathbf{z}_k - \chi_k \tag{45}$$

The controller paramter is selected such that the transient response of both approaches are similar. This is to demonstrate the better steady state performance of the DT-TSMC. In Fig.1-Fig.4, the performance of the DT-TSMC compared to that of the DT-SMC is shown. In Fig.2, it can be seen that the steady state tracking performance of the proposed DT-TSMC approach is superior to that of the classical DT-SMC comtroller for the same control effort as shown in Fig.3. In Fig.4, the disturbance estimation performance of the delay disturbance observer is shown. Finally in Fig.5, the performance of the approach is shown for $p \in [0.1, 1]$. It can be seen that, for this system, the optimal performance is when $p \approx \frac{4.3}{5}$. In conclusion, it is clearly seen that when the transient response of both approaches are matched the steady state performance of the DT-TSMC is better. This is due to the fractional power term which affects the performance of the control when the error is small.

V. Conclusions

In this paper, an outout feedback discrete-time terminal sliding mode control approach for SISO systems was presented. It was shown that the control approach can drive the system to an error of $O\left(T^2\right)$. A rigorous stability proof was presented that shows the fractional power term is effective at the steady state and can improve the tracking performance. Finally, it was demonstrated via a simulation example that the proposed approach can outperform the classical DT-SMC even with similar transient performance.

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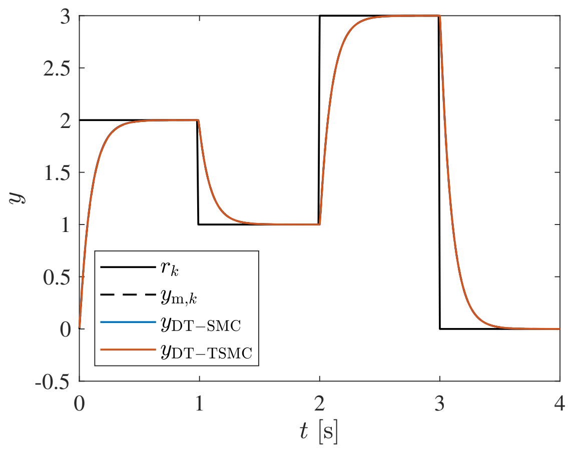

Fig. 1. Output tracking comparison of DT-TSMC and DT-SMC.

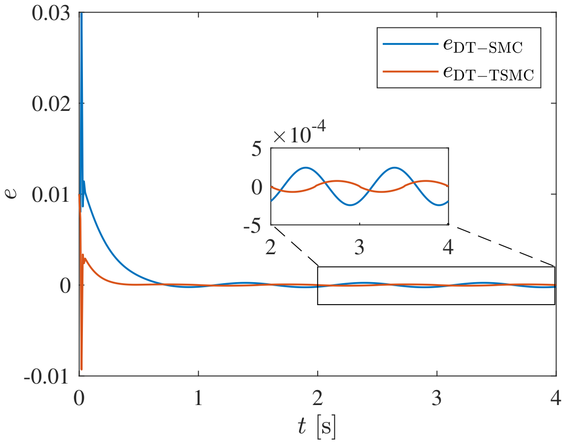

Fig. 2. Output tracking error comparison of DT-TSMC and DT-SMC.

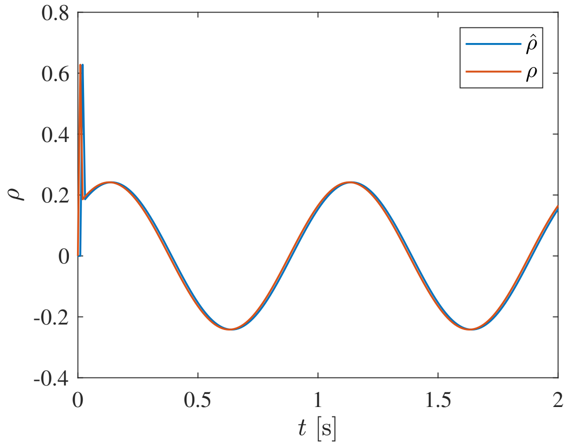

Fig. 4. Performance of the delay disturbance observer approach.

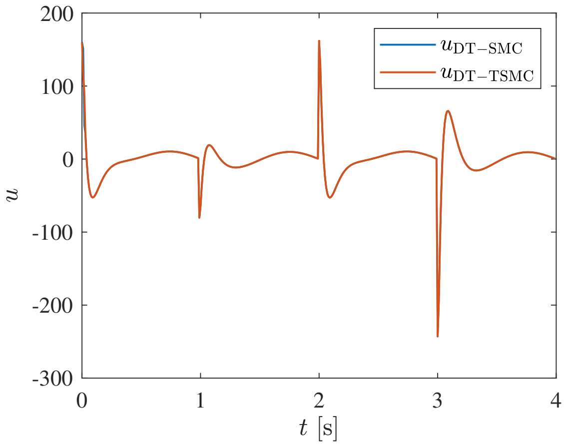

Fig. 3. Control input comparison of DT-TSMC and DT-SMC.

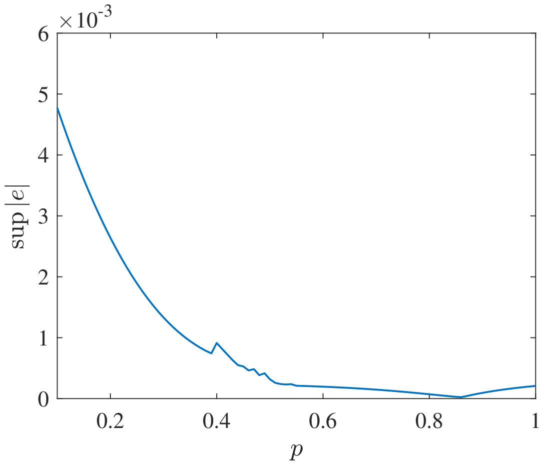

Fig. 5. Maximum steady state error as a function of the exponent p.

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